Fuzzy if-then rules

- Associates a condition described using linguistic variables and fuzzy sets to a conclusion
- A scheme for capturing knowledge that involves imprecision

Two types of fuzzy rules

- Fuzzy mapping rules
  - A functional mapping relationship between inputs and an output using linguistic terms
  - Function approximation in system identification and artificial neural network
- Fuzzy implication rules
  - A generalized logic implication relationship between two logic formula
  - Related to classical two-valued logic and multivalued logic

Fuzzy implication rules

- A generalization of ‘implication’ in two value logic.
- To reason with ideas and statements that are imprecise.

Given: Jack‘s IQ is *High* → Jack is *Smart*.
Jack‘s IQ is *somewhat High*.

Infer: Jack is *somewhat Smart*.

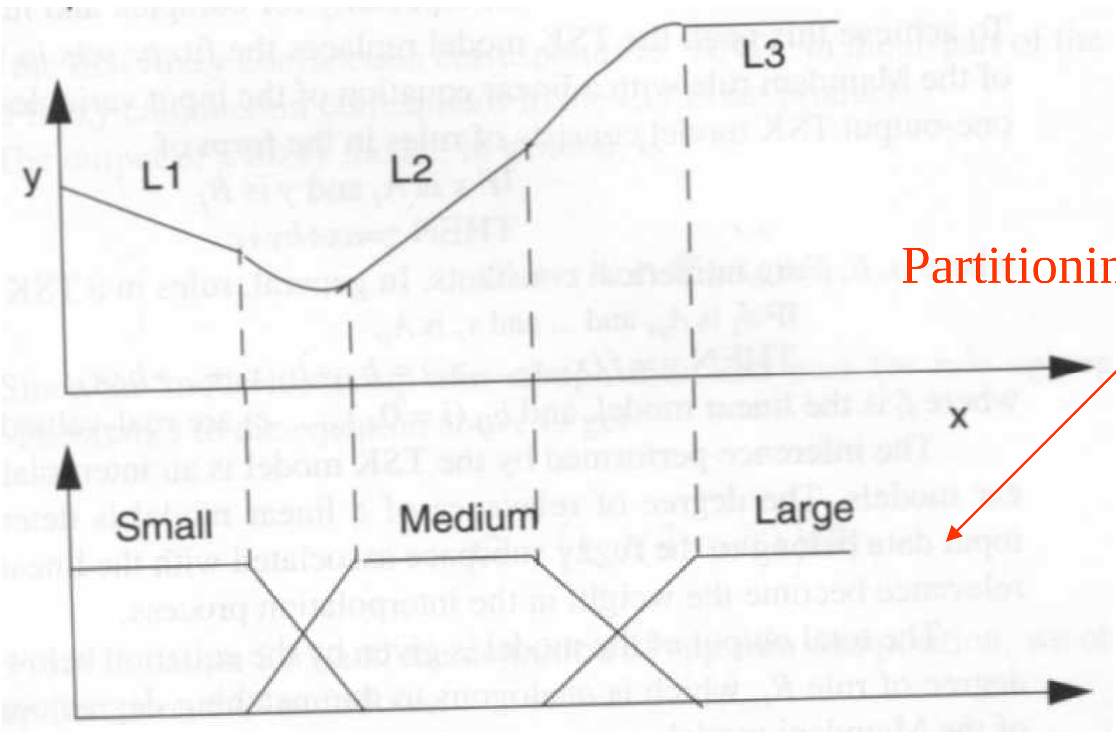
Fuzzy mapping rules

- Function approximation techniques
  - Global modeling
    - Using one mathematical structure
    - Linear, second-order polynomial
  - Local modeling
    - Superimposition functions (B-splines, Taylor expansions)
    - Partition-based approximate
      - Partitioning the input space of the function and approximate the function in each partitioned region

Fuzzy rule-based function approximation

Fuzzy rule-based function approximation

- Fuzzy partition
  - Allowing a sub region to partially overlap with neighboring sub region

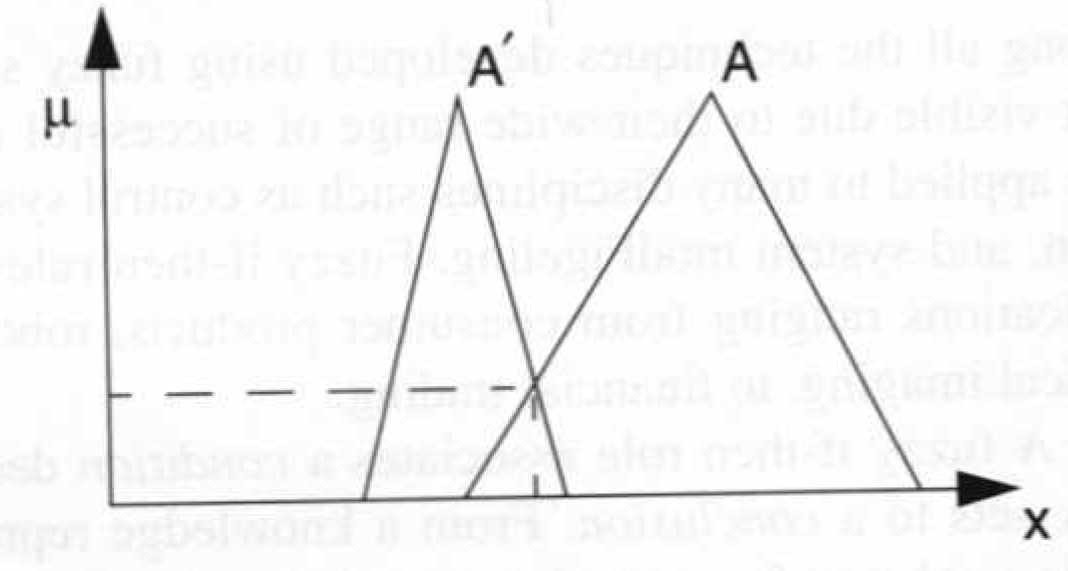


Partitioning the input space of the function

Partial matching

- Calculate the matching degree between a fuzzy input A’ and a fuzzy condition A

$$\text{matchingdegree}\,(A, A') = \sup_{x}\; \min(\mu_A(x), \mu_{A'}(x))$$



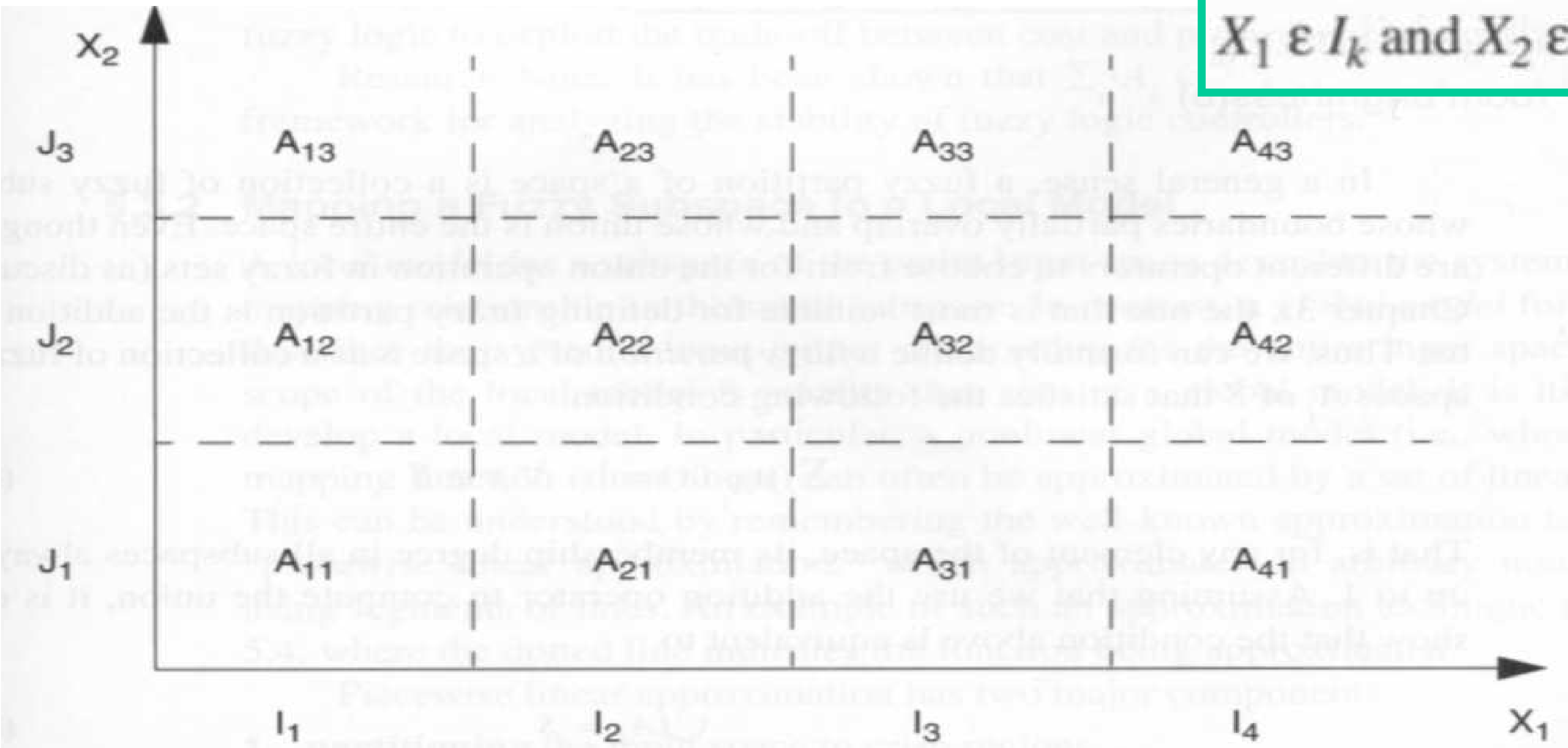
Fuzzy rule-based models

- Fuzzy partition
- Mapping of fuzzy sub regions to local model
- Fusing of multiple local models
- Defuzzification

Classical partition

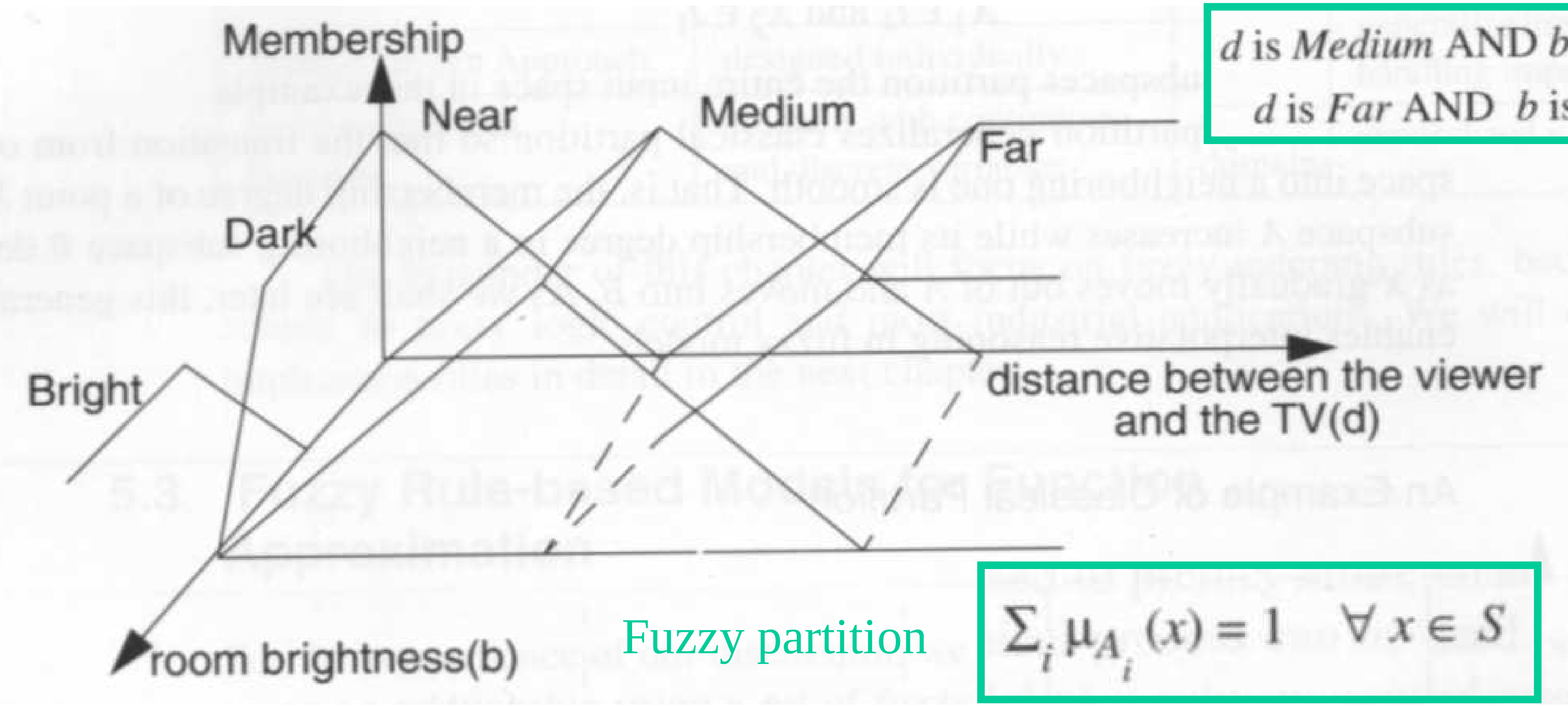
- A classical partition of a space is a collection of disjoint subspaces whose union is the entire space.

$X_1 \varepsilon I_k$ and $X_2 \varepsilon J_1$



Fuzzy partition

- Generalizes classical partition so that the transition from one subspace into a neighbor one is smooth.



d is *Medium* AND *b* is *Dark*.

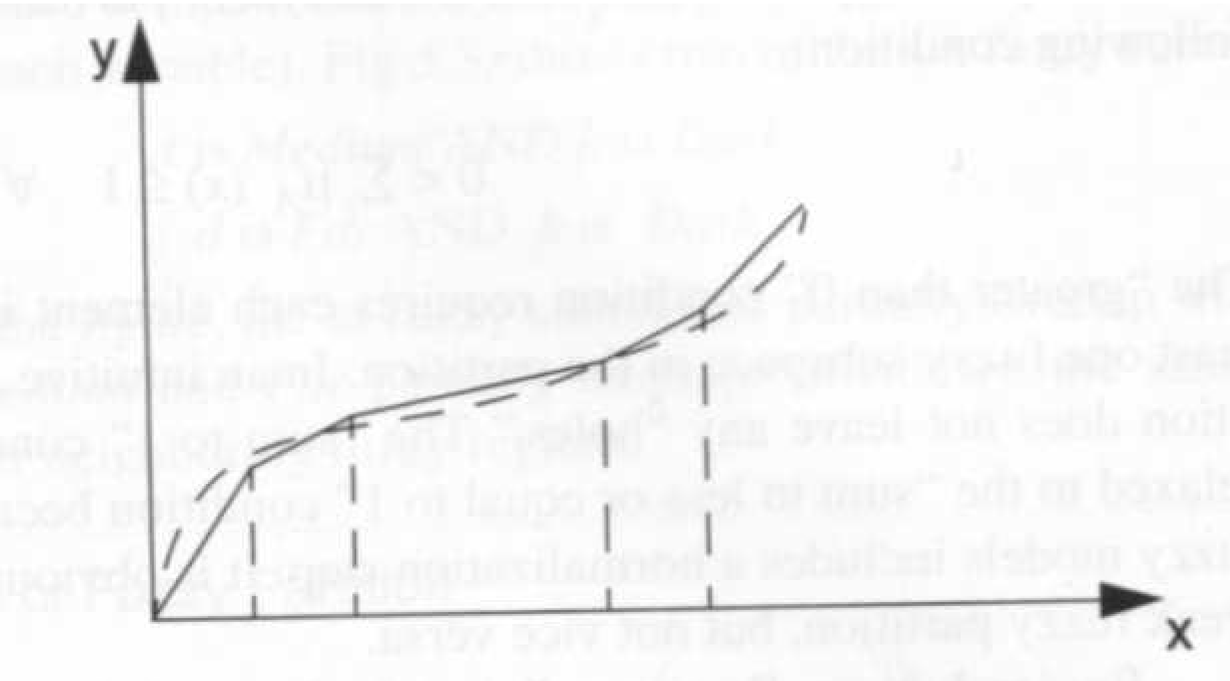
d is *Far* AND *b* is *Dark*

Fuzzy partition

$$\Sigma_i \mu_{A_i}(x) = 1 \quad \forall x \in S$$

Piecewise linear approximation

- A nonlinear global model can often be approximated by a set of linear local models.



Mapping of fuzzy space to local model

- General form

  IF $\vec{x}$ is in FS_i THEN $y_j = LM_i(\vec{x})$

- Four different types of local model
  - Crisp constant

    IF x_2 is Small THEN $y = 4.5$.

  - Fuzzy constant

    IF x is Small THEN y is Medium.

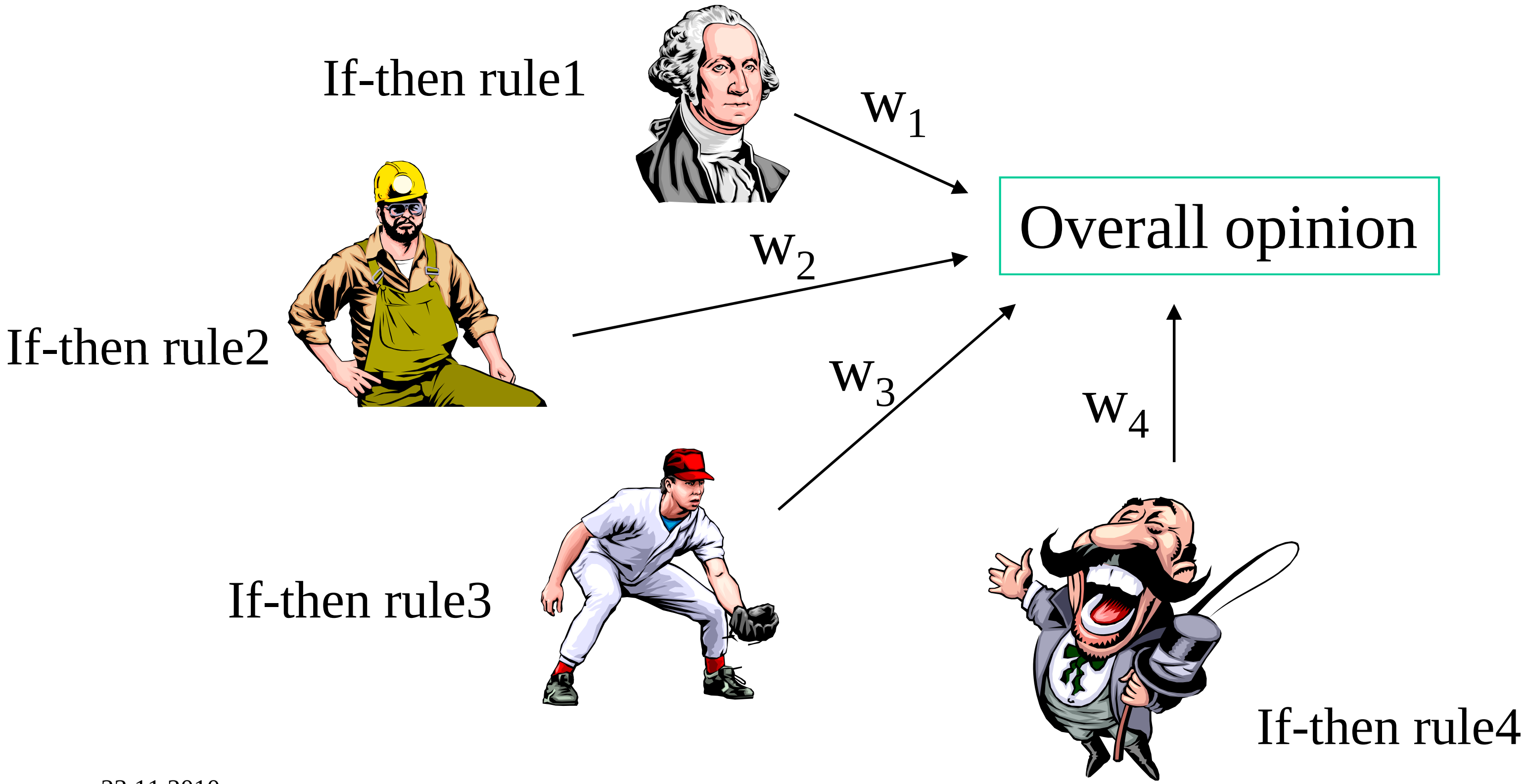
  - Linear model

    IF x is small AND x_2 is Large THEN $y = 2x_1 + 5x_2 + 3$.

  - Nonlinear model

Fusion of local models through interpolative reasoning

- Interpolative reasoning

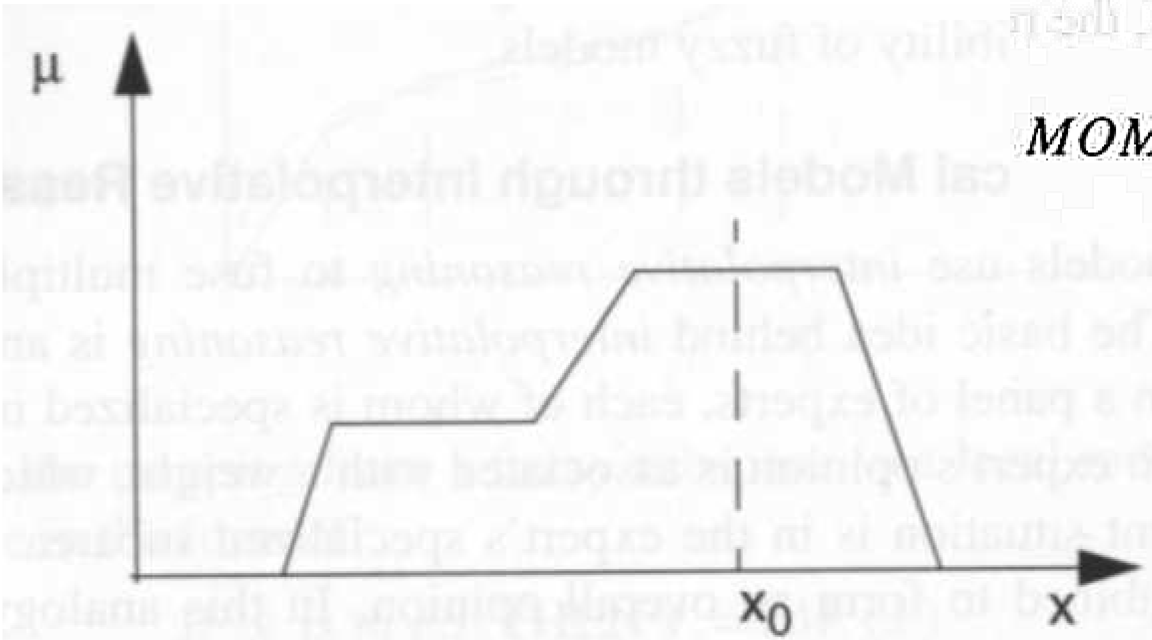


Interpret a possibility distribution

- Linguistic approximation
  - A qualitative interpretation
- Defuzzification
  - A quantitative summary
    - Mean of maximum (MOM)
    - Center of area (COA)
    - The height method

Mean of maximum (MOM)

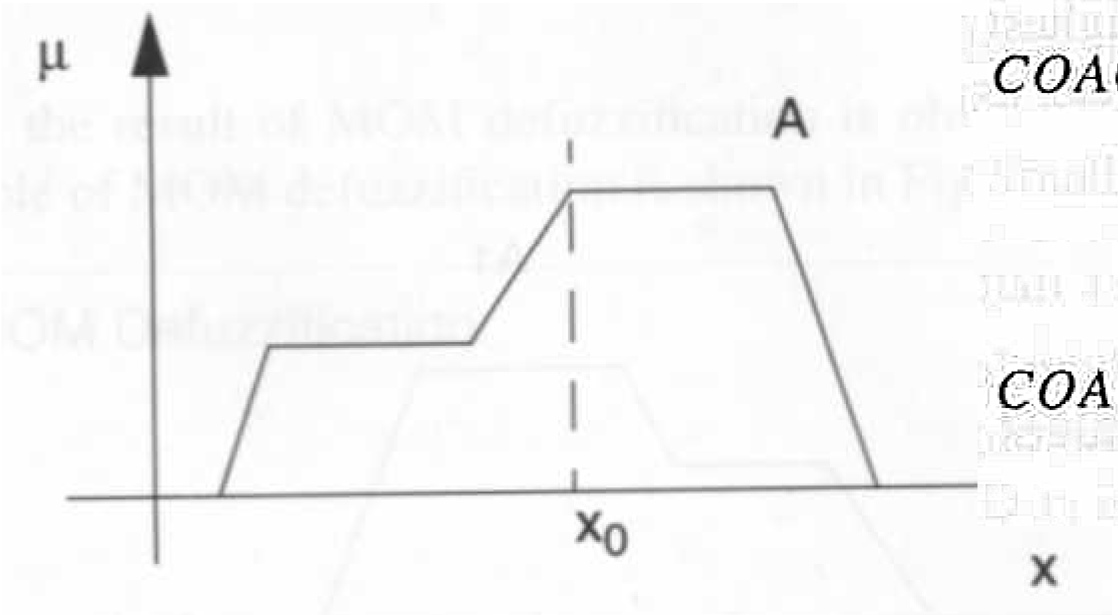
- Calculates the average of those output values that have the highest possibility degrees



$$MOM(A) = \frac{\sum_{y^* \in P} y^*}{|P|}$$

Center of area (COA)

- Calculate the center-of-gravity (the weighted sum of the results)



$$COA(A) = \frac{\sum_x \mu_A(x) \times x}{\sum_x \mu_A(x)}$$

$$COA(A) = \frac{\int \mu_A(x) x dx}{\int \mu_A(x) dx}$$

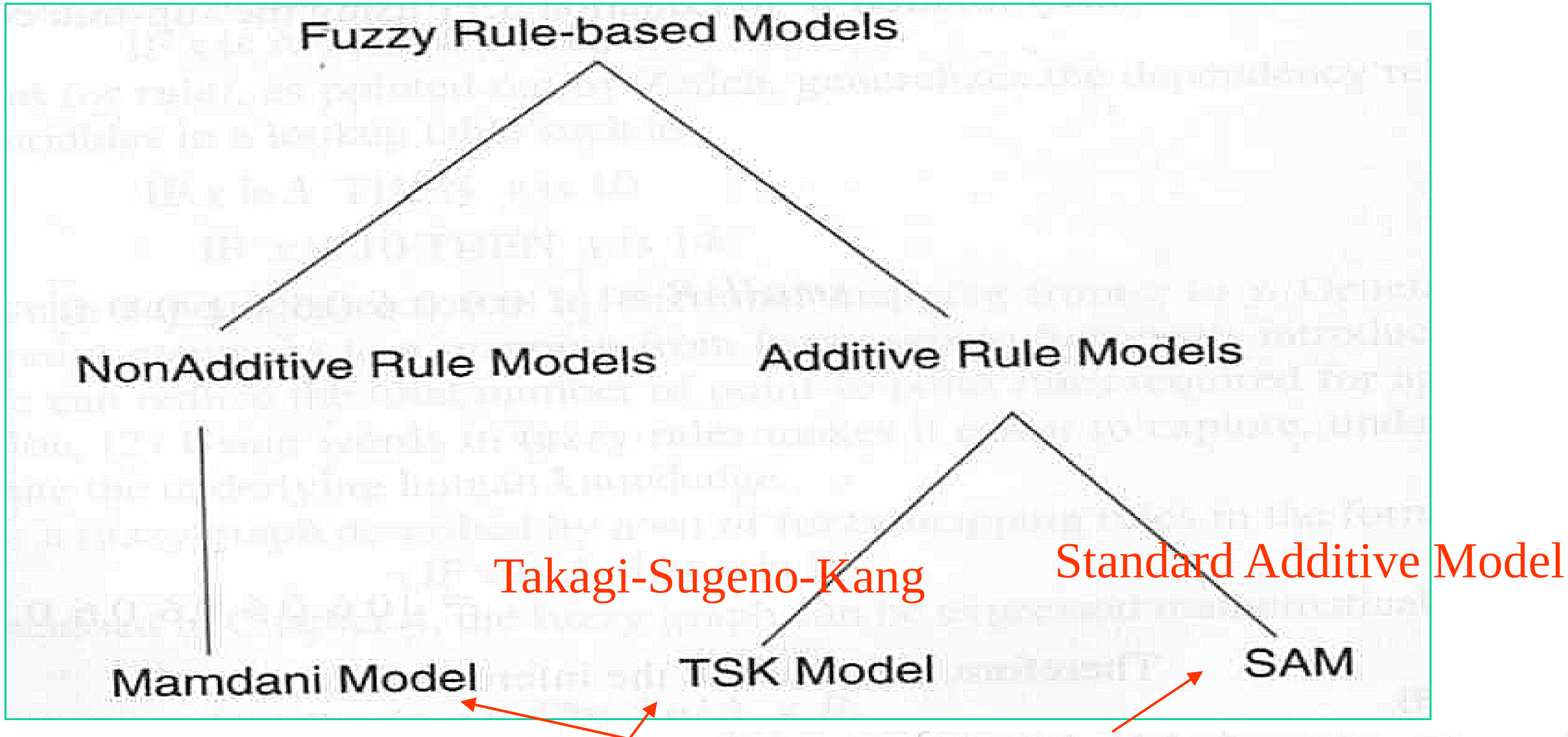
The height method

1. Convert the consequent membership function C_i into crisp consequent $y = c_i$
2. Apply the centroid defuzzification

$$y = \frac{\sum_{i=1}^{M} w_i c_i}{\sum_{i=1}^{M} w_i}$$

w_i is the degree to which the *i*th rule matches the input data

Fuzzy rule-based models



R_i: *If* x_1 is A_{i1} and x_2 is A_{i2} and … and x_s is A_{is}

Then y is C_i, $i = 1, 2, \ldots, M$

R_i: *If* x_1 is A_{i1} and x_2 is A_{i2} and … and x_s is A_{is}

Then $y = f_i(x_1, x_2, \ldots, x_s)$, $i = 1, 2, \ldots, M$

If *speed* is med and *distance* is small then force is negative
If *speed* is zero and *distance* is large then force is positive
…

Mamdani model

Linguistic rules

R_i: IF x_1 is A_{i1} and ... and x_r is A_{ir} THEN y is C_i

Input form

x_1 is A'_1, x_2 is A'_2, ..., x_r is A'_r

output membership of rule i

$$\mu_{C'_i}(y) = (\alpha_{i1} \wedge \alpha_{i2} \wedge \ldots \wedge \alpha_{in}) \wedge \mu_{C_i}(y)$$

Matching degree of rule i, condition j

$$\alpha_{ij} = \sup_{x_j}(\mu_{A'_j}(x_j) \wedge \mu_{A_{ij}}(x_j))$$

Aggregation of outputs from all rules

$$\mu_C(y) = \max\{\mu_{C'_1}(y), \mu_{C'_2}(y), \ldots, \mu_{C'_L}(y)\}$$

TSK model

To reduce the number of rules

Linguistic rules

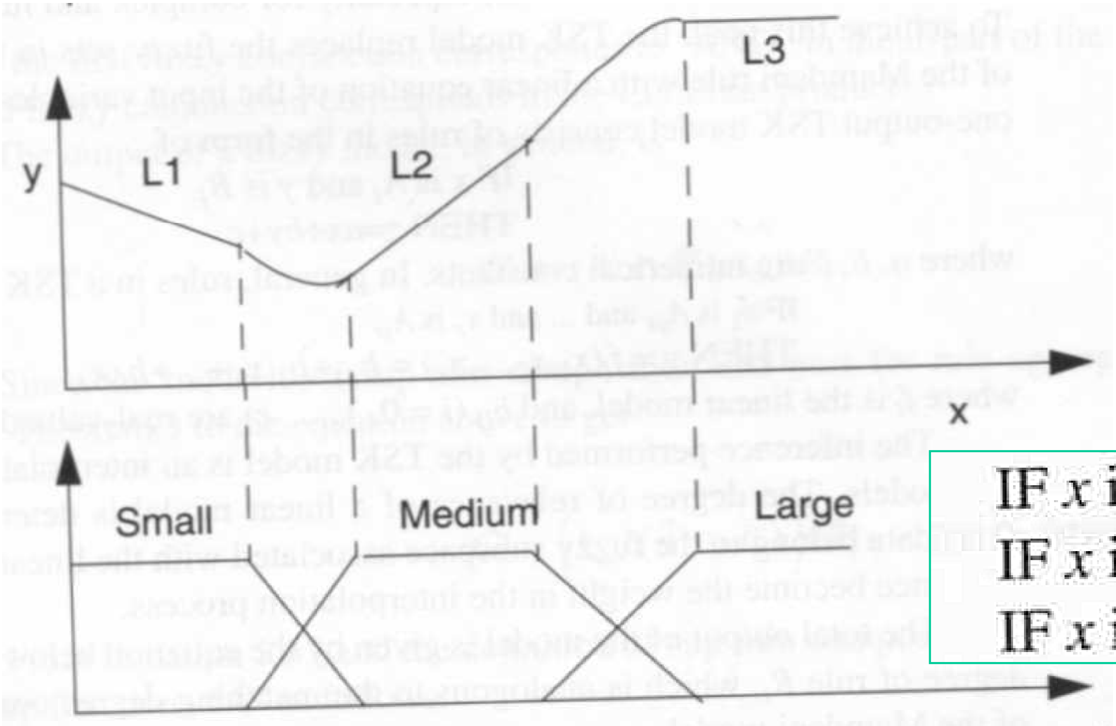
IF x_1 is A_{i1} and ... and x_r is A_{ir}
THEN $y = f_i(x_1, x_2, \ldots, x_r) = b_{i0} + b_{i1}x_1 + \ldots + b_{ir}x_r$

output

$$y = \frac{\sum_{i=1}^{L} \alpha_i f_i(x_1, x_2, \ldots, x_r)}{\sum_{i=1}^{L} \alpha_i} = \frac{\sum_{i=1}^{L} \alpha_i (b_{i0} + b_{i1}x_1 + \ldots + b_{ir}x_r)}{\sum_{i=1}^{L} \alpha_i}$$

$$\alpha_i = \min(\mu_{A_{i1}}(a_1), \mu_{A_{i2}}(a_2), \ldots, \mu_{A_{ir}}(a_r))$$

TSK model (example)



IF x is *Small* THEN $y = L1(x)$.
IF x is *Medium* THEN $y = L2(x)$.
IF x is *Large* THEN $y = L3(x)$.

$$y = \frac{\mu_{small}(x) \times L1(x) + \mu_{medium}(x) \times L2(1)(x) + \mu_{large}(x) \times L3(x)}{\mu_{small}(x) + \mu_{medium}(x) + \mu_{large}(x)}$$

Standard Additive Model

Linguistic rules: R_i: IF x_1 is A_{i1} and ... and x_r is A_{ir} THEN y is C_i

Output

$$z = Centroid\left(\sum_i \mu_{A_i}(x_0) \times \mu_{B_i}(y_0) \times \mu_{C_i}(z)\right)$$

$$= \frac{\sum_{i=1}^{n} (\mu_{A_i}(x_0) \times \mu_{B_i}(y_0)) \times A_i \times g_i}{\sum_{i=1}^{n} (\mu_{A_i}(x_0) \times \mu_{B_i}(y_0)) \times A_i}$$

$$A_i = \int \mu_{C_i}(z)dz$$

$$g_i = \frac{\int z \times \mu_{C_i}(z)dz}{\int \mu_{C_i}(z)dz}$$

A_i is the area under the ith. rule's conclusion C_i and g_i is the centroid of C_i

Comparison of SAM and Mamdani

	SAM	Mamdani
inputs	crisp	Crisp & fuzzy
Composition operator	scaling	Clipping (min)
Fusion method	addition	max
Defuzzification	Centroids	Not insist

Multiconditional Approximate Reasoning

Rule 1 :	If $\mathcal{X}$ is A_1, then $\mathcal{Y}$ is B_1
Rule 2 :	If $\mathcal{X}$ is A_2, then $\mathcal{Y}$ is B_2
....	
Rule n :	If $\mathcal{X}$ is A_n, then $\mathcal{Y}$ is B_n
Fact :	$\mathcal{X}$ is A'
Conclusion :	$\mathcal{Y}$ is B'

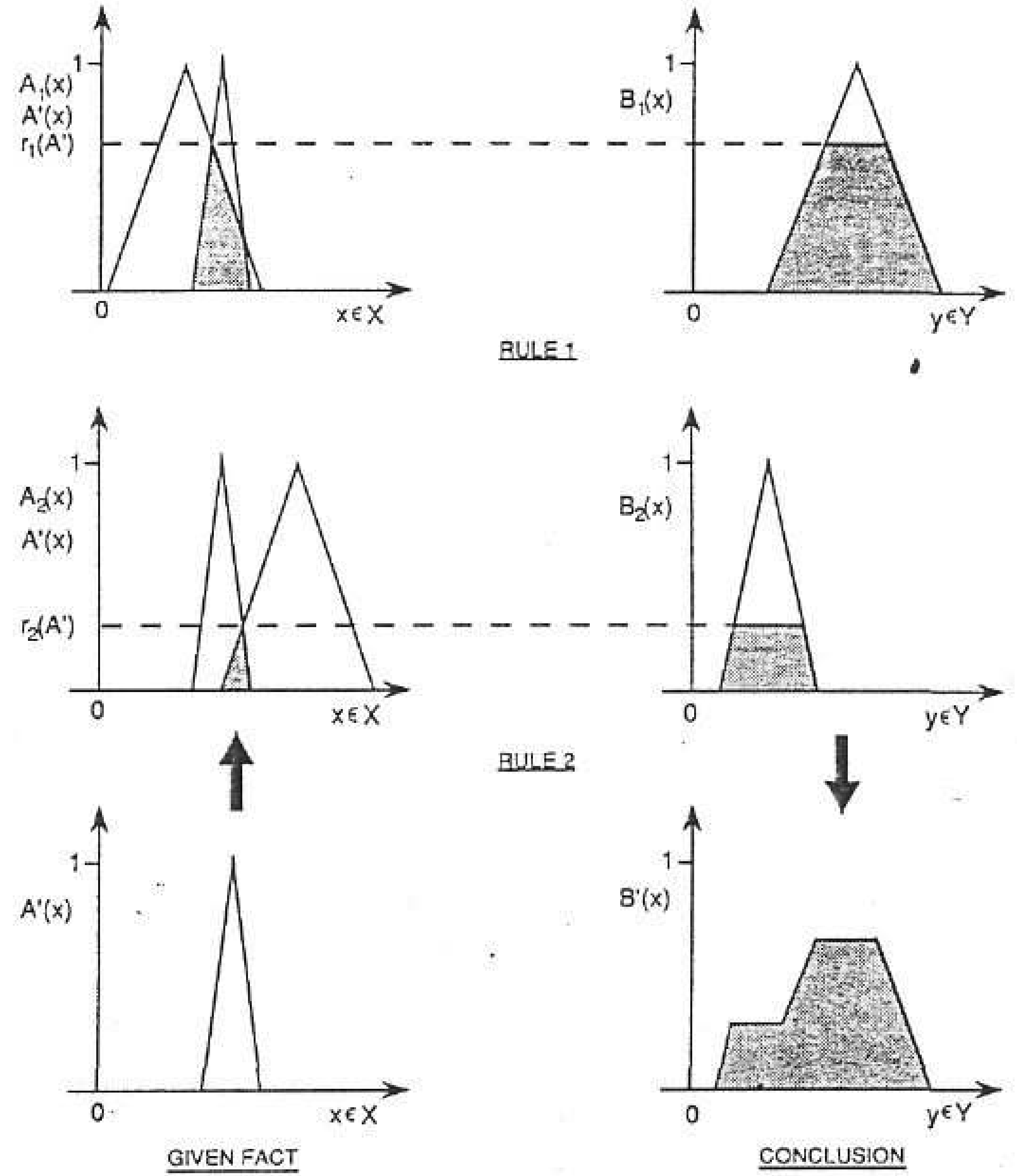
$$R(x, y) = \sup_{j \in \mathbb{N}_n} \min[A_j(x), B_j(y)]$$

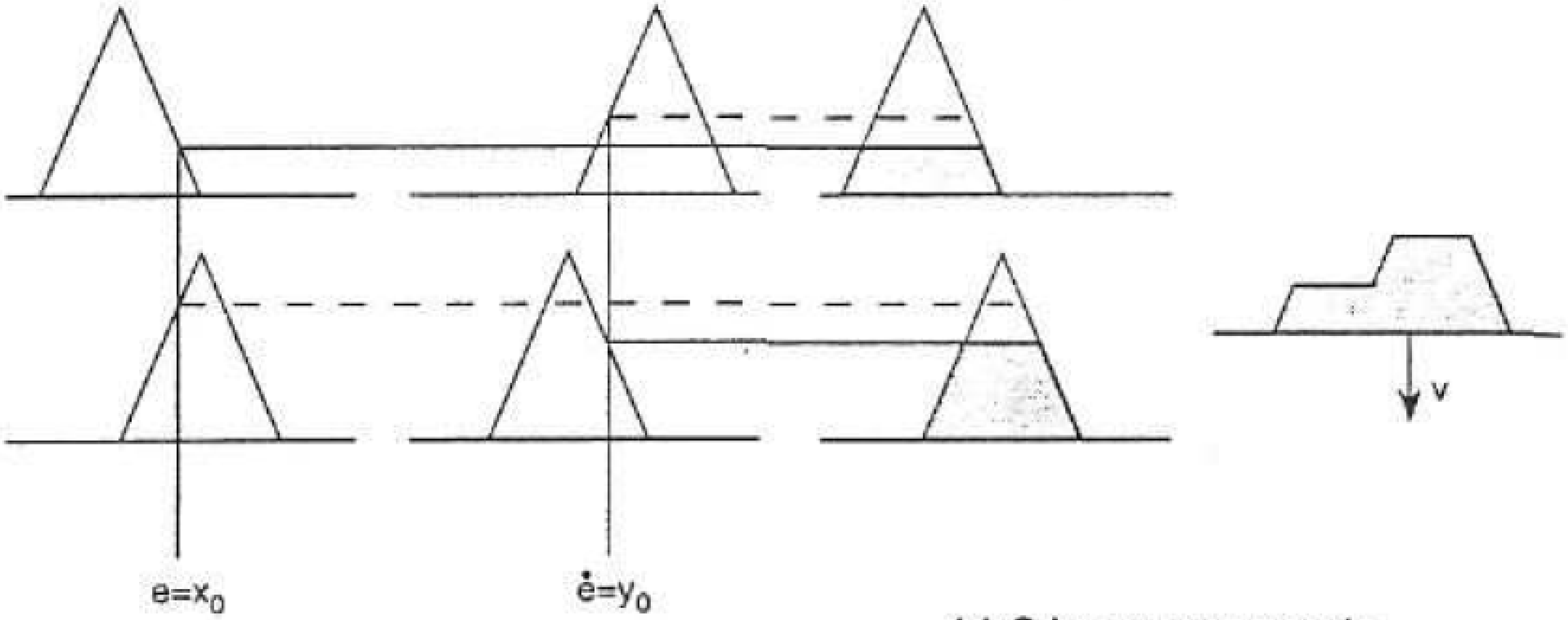
Step 1.

$$r_j(A') = h(A' \cap A_j)$$

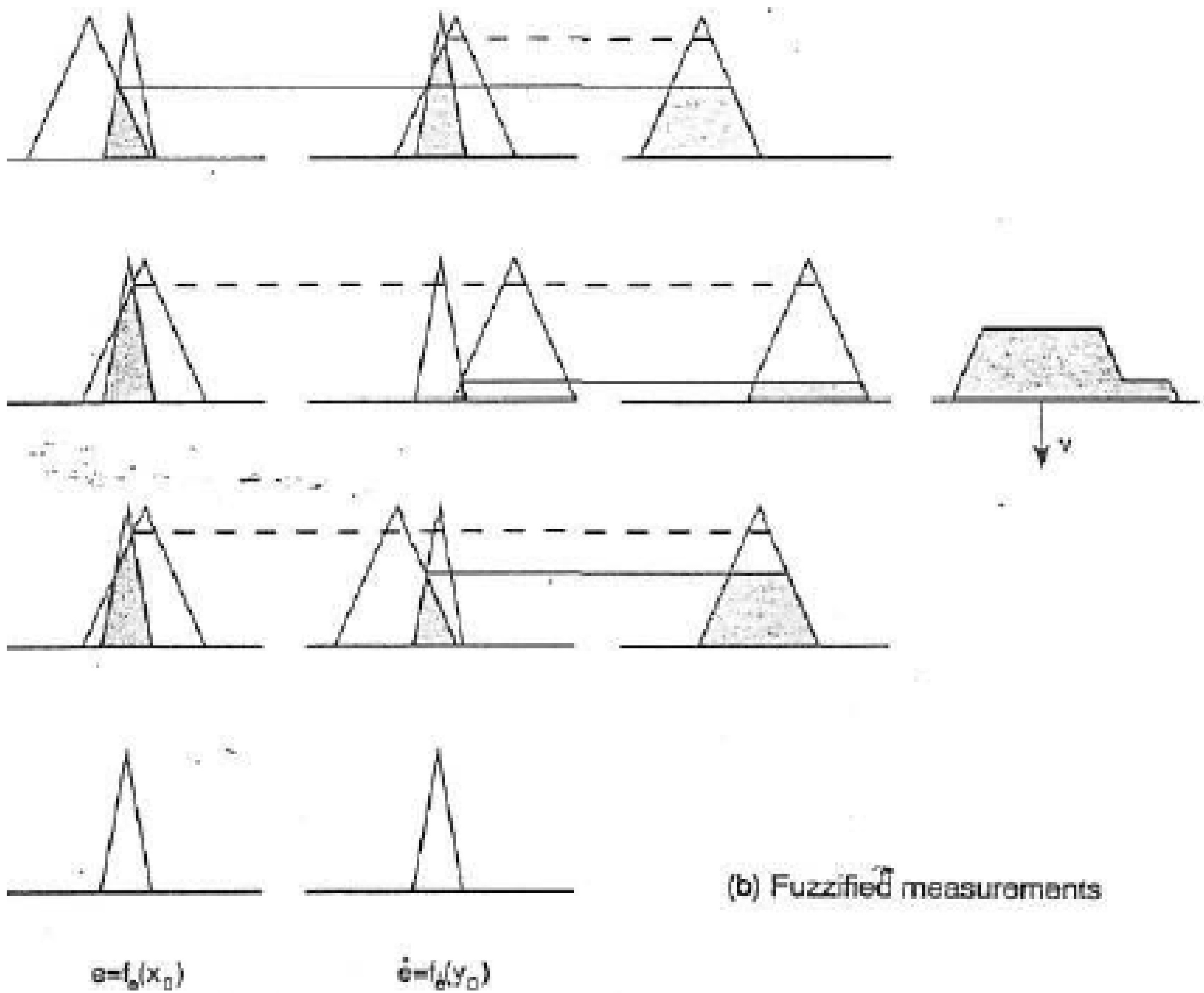
Step 2.

$$B'(y) = \sup_{j \in \mathbb{N}_n} \min[r_j(A'), B_j(y)]$$


1
A1(x)
A'(x)
r1(A')
0
x∈X
1
B1(x)
0
y∈Y
RULE 1
1
A2(x)
A'(x)
r2(A')
0
x∈X
1
B2(x)
0
y∈Y
RULE 2
1
A'(x)
0
x∈X
GIVEN FACT
1
B'(x)
0
y∈Y
CONCLUSION



(a) Crisp measurements



(b) Fuzzified measurements