# Fuzzy logic

- Broad sense

  a system of concepts, principles, and method for dealing with modes of reasoning that are approximate rather than exact.
- Narrow sense

  A generalization of the various multivalued logics

# Multivalued logics

- Take into account the uncertainty of truth values
- Three-valued logics

| $p$ | $\neg p$ |
|---|---|
| 0 | 1 |
| 1/2 | 1/2 |
| 1 | 0 |

- *n*-valued logics

$$T_n = \left\{ \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} \right\}$$

# Connectives of some three-valued logics

| | | Łukasiewicz | | | | Bochvar | | | | Kleene | | | | Heyting | | | | Reichenbach | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *a* | *b* | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Approximate reasoning

Old coins are usually rare collectibles
Rare collectibles are expensive

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∴ Old coins are usually expensive

- Types of fuzzy linguistic terms
  - Fuzzy predicates: tall, young, small, median
  - Fuzzy truth values: true, false, very true
  - Fuzzy probabilities: likely, unlikely, very likely
  - Fuzzy quantifiers: many, few, most

# Fuzzy propositions

- Example:
  - Mount Washington is a dangerous mountain
  - ‘Mount Washington is a dangerous mountain’ is true

Qualified proposition

- Conditional propositions
  - assertions that are in conditional if-then form
  - Example: if Tina is young, then John is old
- Qualified propositions
  - Propositions that are asserted to be simply true.

# Unconditional and unqualified propositions

The temperature

35ºC is high.

- Propositional form
  - $p$: $\chi$ is A

$\chi$ is a variable

A is some property or predicate

$p$:' $\chi$ is A' is true

- $T(p_x)$ = the degree of truth of $p_x$
  - $p_x$: $\chi = x$ is A

$T(p_x) = A(x)$

The degree of $x$ belong to $\chi$

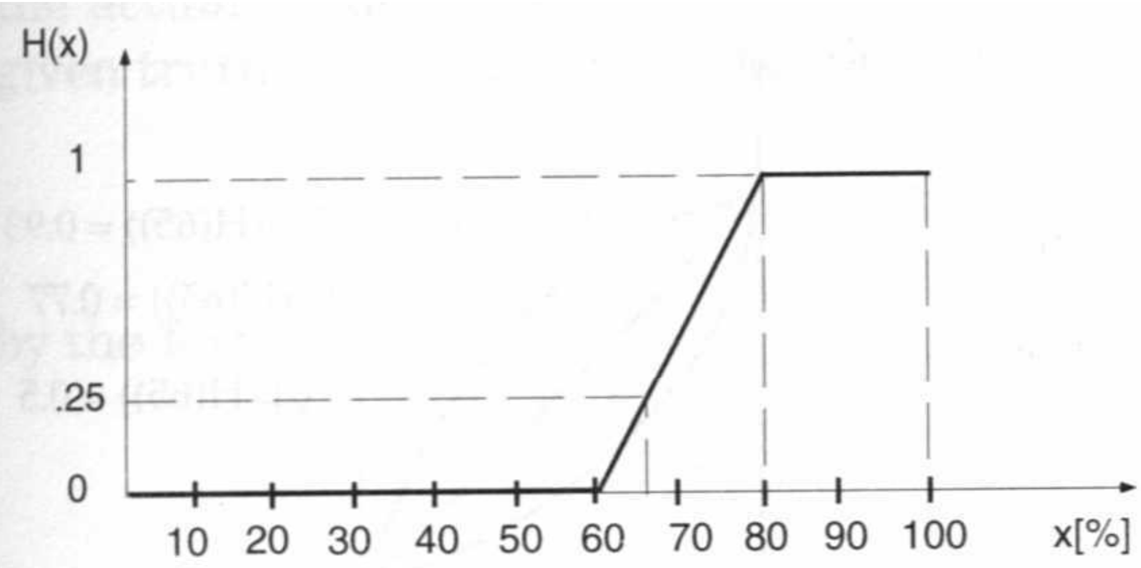
# Example

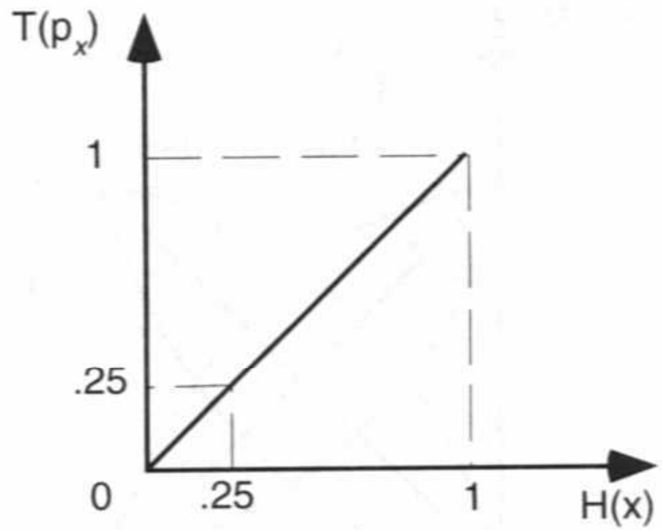
$p_{65}$: Humidity of 65% is high

The degree of $p_{65}$ is true is
$T(p_{65}) = H(65) = 0.25$

$T(p_x) = H(x)$

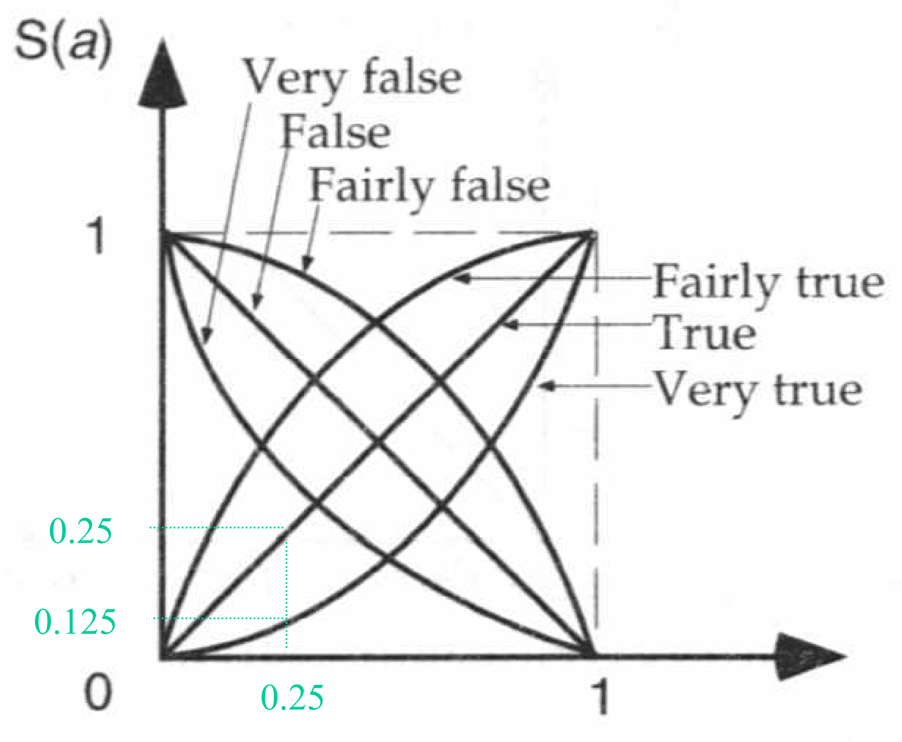
The degree of $x$ belong to $\chi$



# example

- $p_{65}$: 'Humidity of 65% is high' is *very true*




The degree of truth of $p_{65}$ is

$T_s(p_{65}) = S(A(x)) = S(0.25) = 0.125$

# Unconditional and qualified propositions

- Propositional form
  - $p$: ‘$\chi$ is A’ is $S$

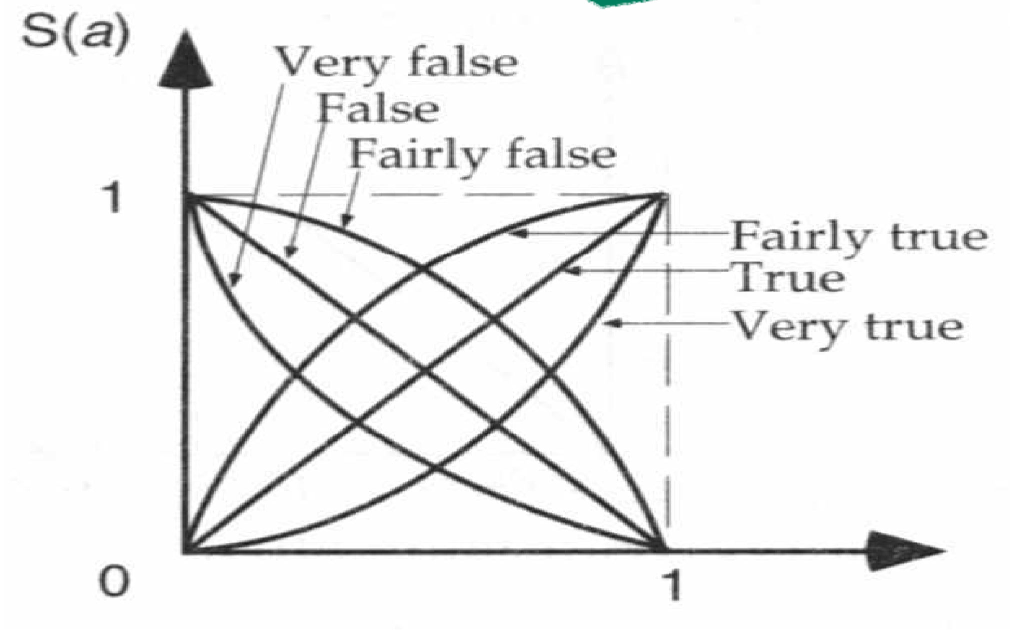
‘Humidity of 65% is high’ is *very false*

*S* is a fuzzy truth qualifier

The degree of truth, $T_s(p_x)$ of the qualified proposition

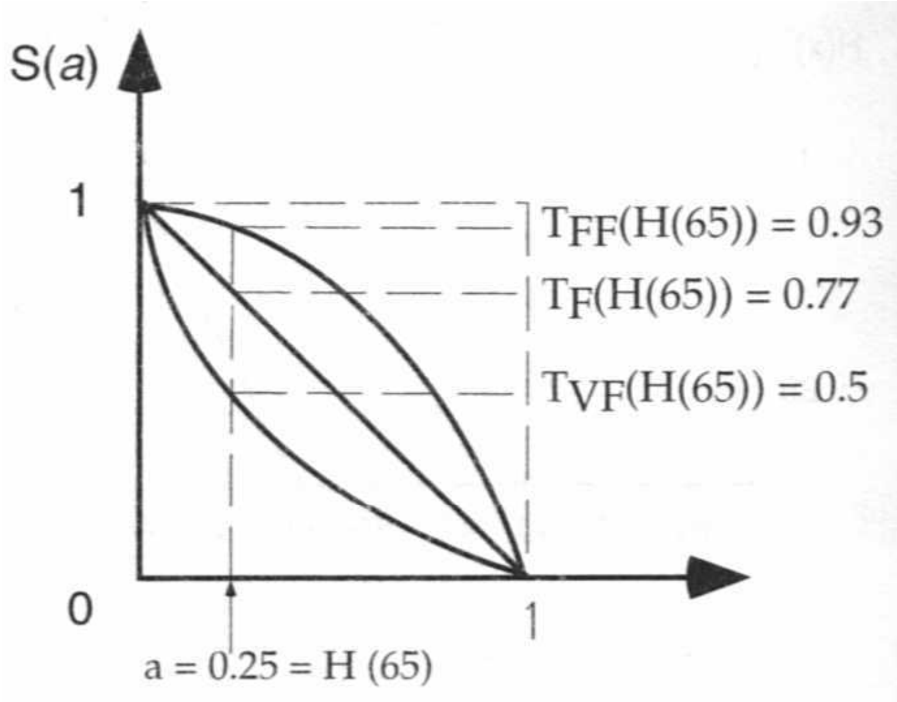
$p_x$:‘$\chi = x$ is A’ is $S$

is $T_s(p_x) = S(A(x))$



# example

- $p_{65}$: 'Humidity of 65% is high' is *very false*




The degree of truth of $p_{65}$ is
$T_s(p_{65}) = S(A(x)) = S(0.25) = 0.5$

# Conditional and unqualified propositions

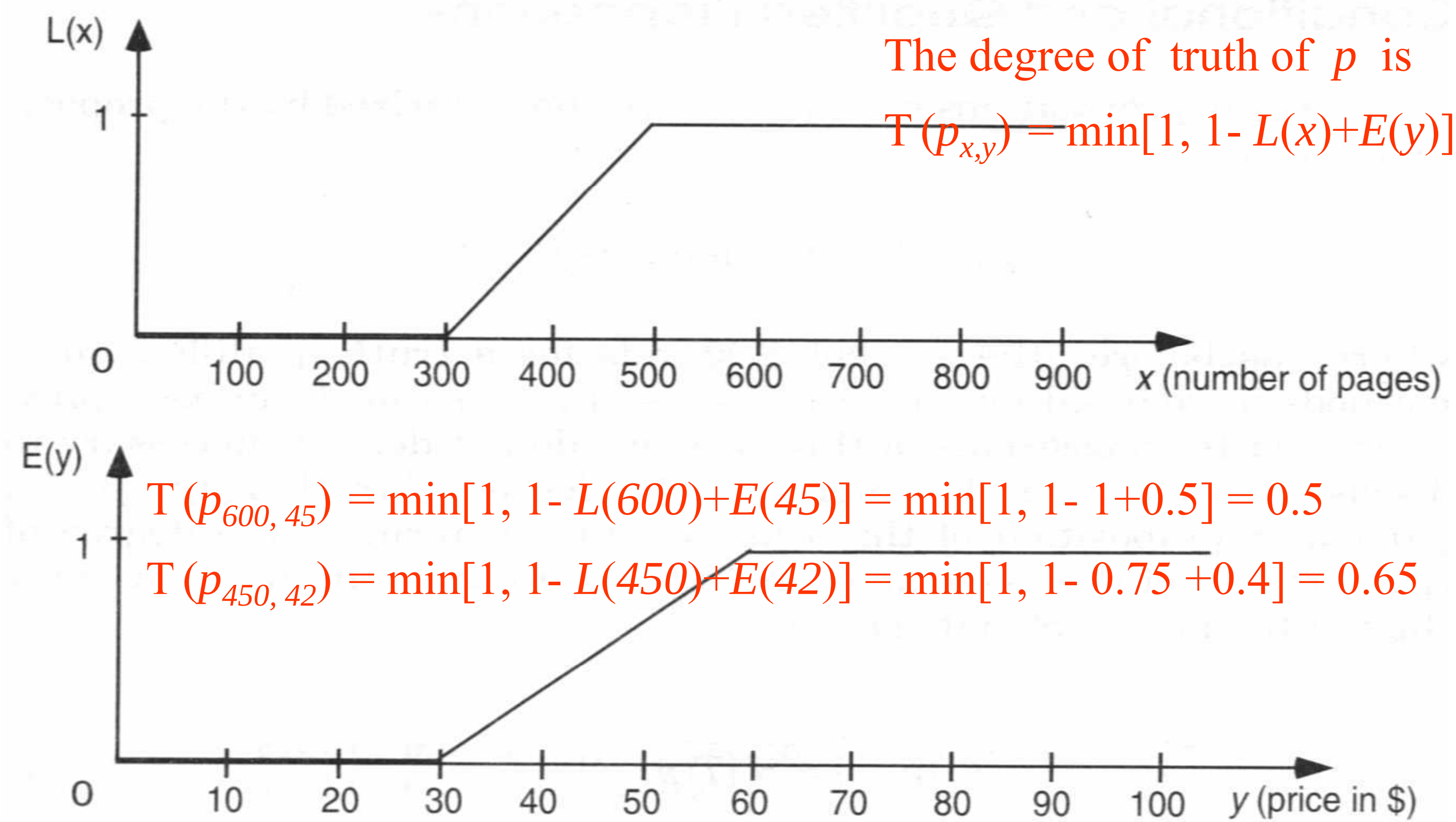
- Propositional form  $\chi$ is A  $\gamma$ is $B$
  - $p$: if $\chi$ is A, then $\gamma$ is $B$
  - $p_{x,y}$: 'if A($x$), then $B$($y$)' is true
  - Fuzzy implication A($x$) $\Rightarrow$ $B$($y$)
- The degree of truth
  - $T(p_{x,y}) = I[A(x), B(y)] = \min[1, 1- A(x)+B(y)]$

Lukasiewicz implication

If Tina is young, then John is old

# example

- *p*: if a textbook is large , then it is expensive




The degree of truth of *p* is

$T(p_{x,y}) = \min[1, 1- L(x)+E(y)]$

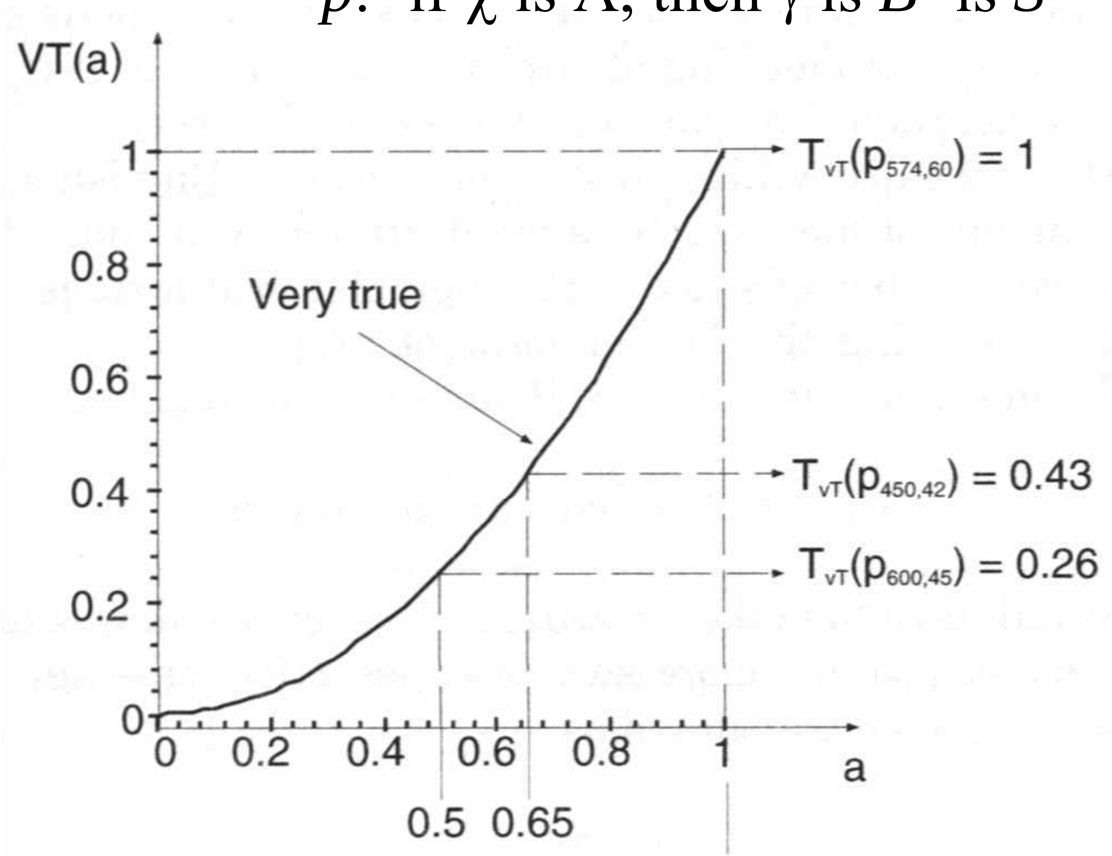
$T(p_{600,\,45}) = \min[1, 1- L(600)+E(45)] = \min[1, 1- 1+0.5] = 0.5$

$T(p_{450,\,42}) = \min[1, 1- L(450)+E(42)] = \min[1, 1- 0.75 +0.4] = 0.65$

# Conditional and qualified propositions

- Propositional form
  - $p$: 'if $\chi$ is A, then $\gamma$ is $B$' is $S$

'If a textbook is large, then it is expensive' is very true




The degree of truth

$T_s(p_{x,y}) = S[T(p_{x,y})]$

# Fuzzy quantifiers

Bazı (En az bir tane var)

Her

- Two quantifiers of predicate logic
  - Universal quantifier: *all*, $\forall$
  - Existential quantifier: *there exist*, $\exists$
- Fuzzy quantifiers
  - *Absolute quantifiers*
    - About a dozen, at most about 10, at least about 100
    - About 20 hotels are in close proximity to the center of the city
  - *Relative quantifiers*
    - Most, almost all, about half, about 20%
    - Almost all hotels are in close proximity to the center of the city

All snakes are reptiles
$(\forall x)\ (\ Sx \Rightarrow Rx)$
Some snakes are poisonous
$(\exists x)\ (\ Sx \wedge Px)$
Almost all snakes are poisonous

# Linguistic hedges

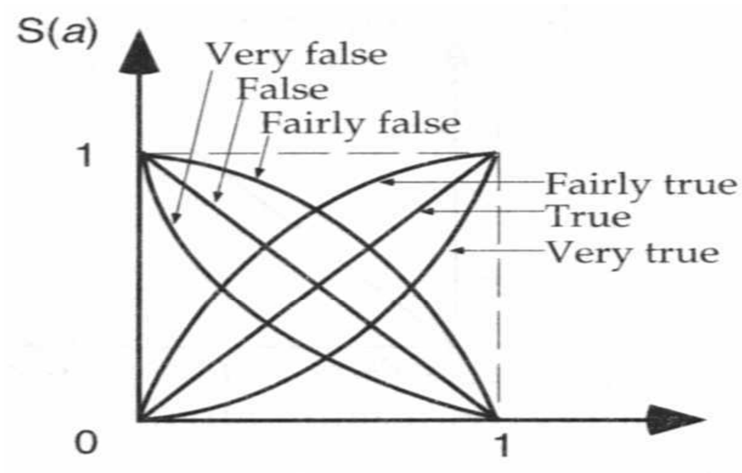
- Special linguistic terms by which other linguistic terms are modified.
  - Very, more ,less, fairly, extremely
- Modifier
  - $HA(x) = h(A(x))$
  - $h(a) = a^{1/2}$ , $a^2$, $a^3$, …

All mountains are steep
Almost All mountains are steep

Weak modifier, fairly

strong modifier, very

Very strong modifier, very very



# Example

- Propositions
  - $p_1$: John is young
  - $p_2$: John is very young
  - $p_3$: John is  fairly young
- Assume John is 26 years old, the degree of truth of the propositions are
  - *Young*(26) = 0.8
  - *Very young*(26) = $0.8^2$ = 0.64 ← Strong assertion is less true
  - *Fairly young* (26) = $0.8^{1/2}$ = 0.89

# Approximately reasoning

- Deductive reasoning
  - Use valid argument form
- Approximately reasoning
  - Dealing with reasoning under fuzzy environment

*Modus Ponens (MP)*

1. $p \Rightarrow q$
2. $p$

$\therefore \quad q$

$$[(p \Rightarrow q) \wedge p] \Rightarrow q$$

| | |
|---|---|
| Rule: | If a book is large, then it is expensive |
| Fact: | Book $x$ is fairly large |
| Conclusion: | Book $x$ is fairly expensive |

mantıkta;
"a olursa b olur."
"a olmuştur."
önermelerinden
"b de olmuştur."
sonucunu çıkarmaya
verilen isim.

# Generalized modus ponens

| | |
|---|---|
| Rule: | If $\mathcal{X}$ is $A$, then $\mathcal{Y}$ is $B$ |
| Fact: | $\mathcal{X}$ is $A'$ |
| Conclusion: | $\mathcal{Y}$ is $B'$ |

$$B'(y) = \sup_{x \in X} \min \left(A'(x), I(A(x), B(y))\right)$$