Fuzzy relations

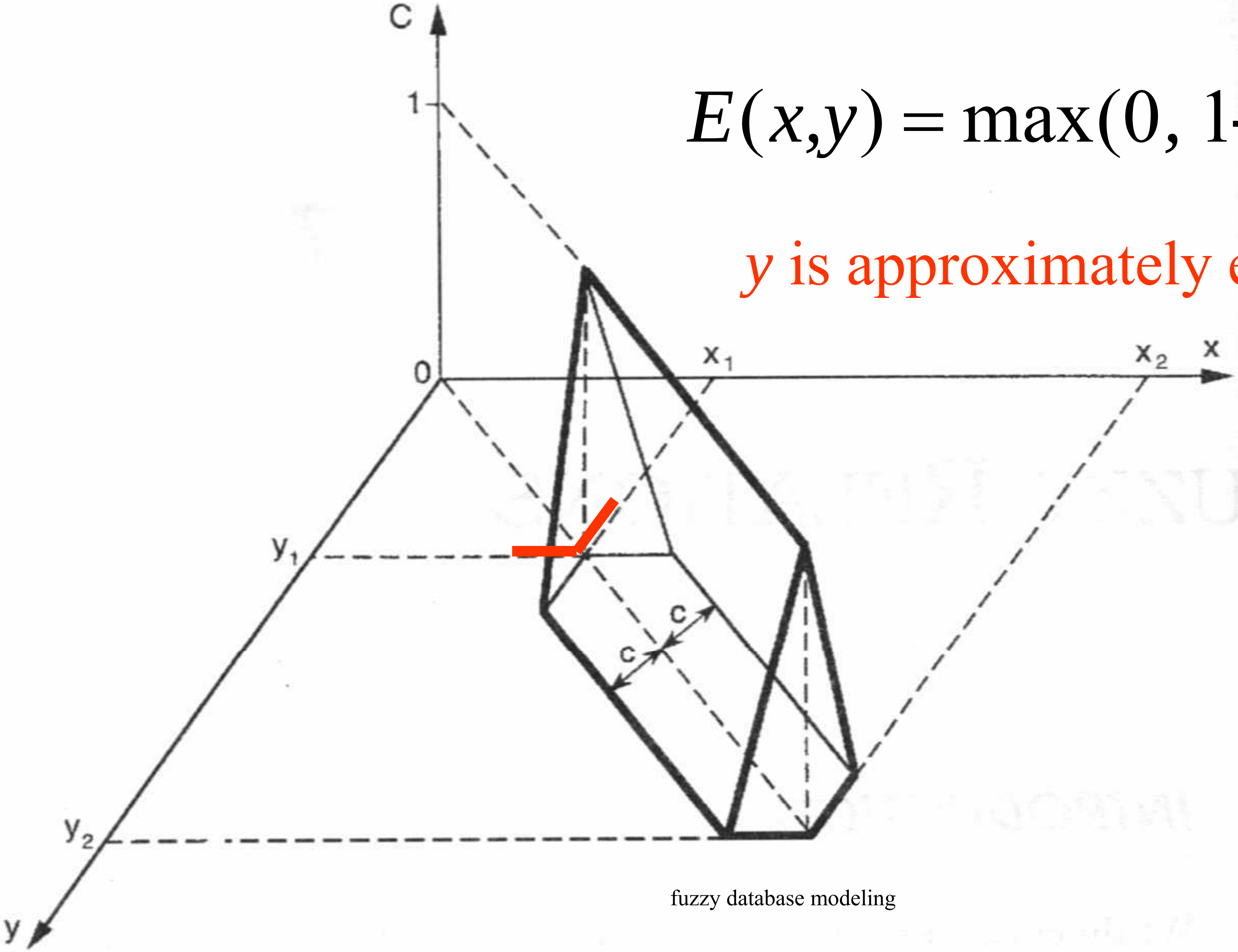
Fuzzy sets defined on universal sets which are Cartesian products.

<x, y>

Example (information retrieval, bilgi erişimi)

Membership degree R(d,t):
the degree of relevance of document d and the key term t

Example (equality relation)



$$E(x,y) = \max(0,\ 1-\frac{|x-y|}{c})$$

y is approximately equal to *x*

Representations of fuzzy relation

- List of ordered pairs with their membership grades
- Matrices
- Mappings
- Directed graphs

Matrices

Fuzzy relation R on X×Y

$X=\{x_1, x_2, \ldots, x_n\}$, $Y=\{y_1, y_2, \ldots, y_m\}$

$r_{ij}=R(x_i, y_j)$ is the membership degree of pair (x_i, y_j)

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix}$$

Example (very far from)

Example: X={Beijing, Chicago, London, Moscow, New York, Paris, Sydney, Tokyo}

R	*B*	*C*	*L*	*M*	*N*	*P*	*S*	*T*
B	0	1	0.7	0.5	1	0.7	0.6	0.1
C	1	0	0.5	0.9	0	0.5	1	1
L	0.7	0.5	0	0.3	0.5	0	1	0.7
M	0.5	0.9	0.3	0	0.9	0.3	0.8	0.5
N	1	0	0.5	0.9	0	0.5	1	1
P	0.7	0.5	0	0.3	0.5	0	1	0.7
S	0.6	1	1	0.8	1	1	0	0.6
T	0.1	1	0.7	0.5	1	0.7	0.6	0

Very far from

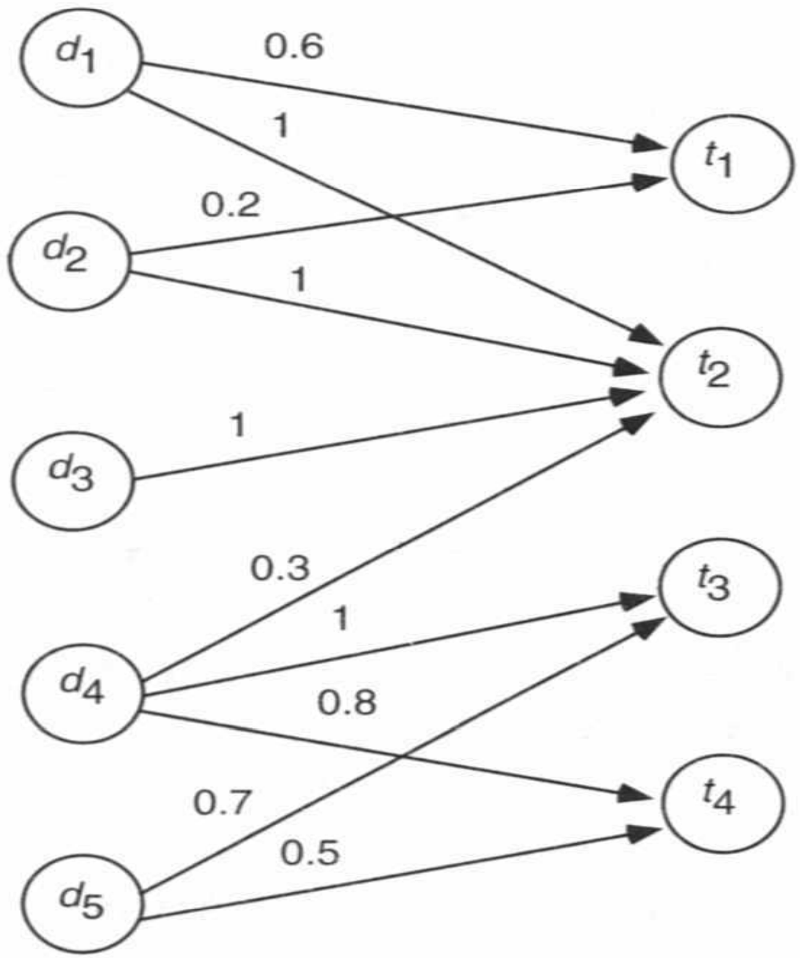
Mappings (Eşlemeler)

The visual representations

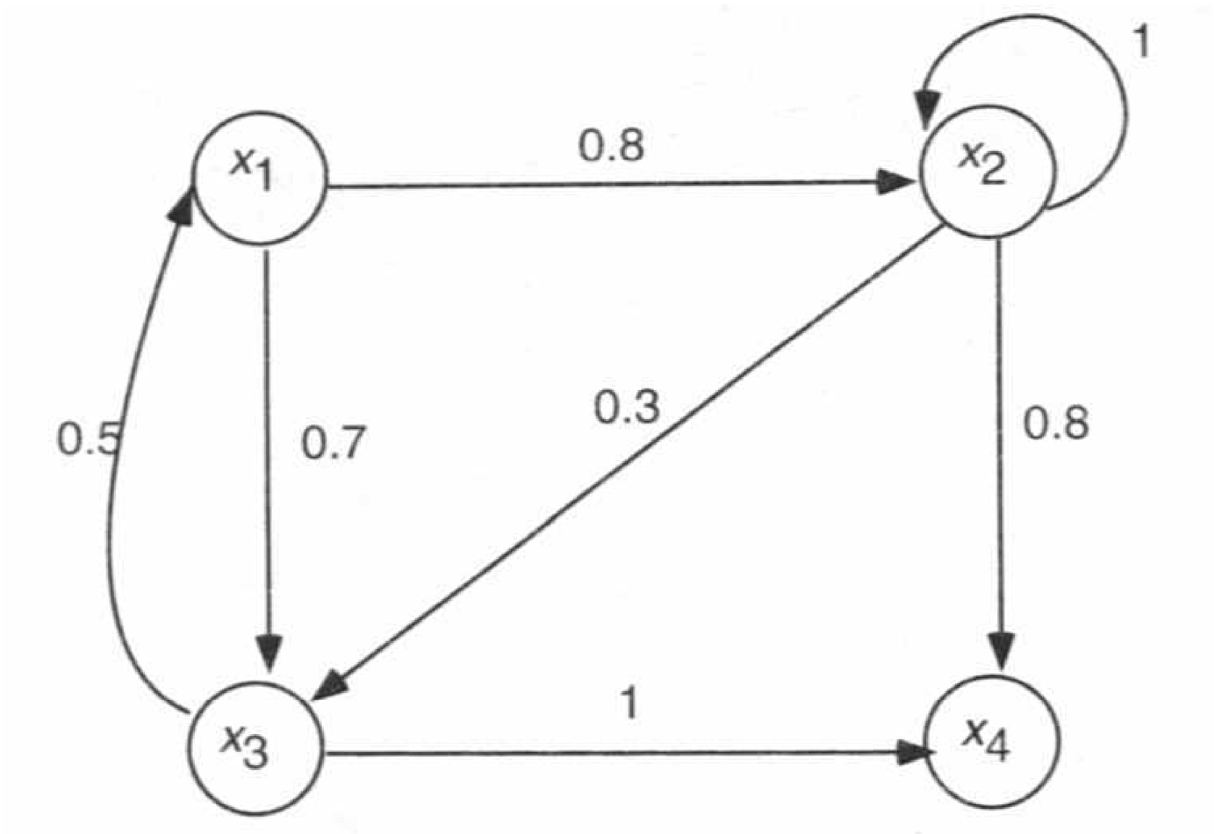
On finite Cartesian products

Document D = $\{d_1, d_2, \ldots, d_5\}$

Key terms T = $\{t_1, t_2, t_3, t_4\}$



Directed graphs



Inverse operation

- Given fuzzy binary relation $R \subseteq X \times Y$
  - Inverse $R^{-1} \subseteq Y \times X$
  - $R^{-1}(y,x) = R(x,y)$
  - $(R^{-1})^{-1} = R$

The transpose of R

$$\mathbf{R} = \begin{bmatrix} 0.6 & 1 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.3 & 1 & 0.8 \\ 0 & 0 & 0.7 & 0.5 \end{bmatrix} \quad \text{and } \mathbf{R}^{-1} = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0.3 & 0 \\ 0 & 0 & 0 & 1 & 0.7 \\ 0 & 0 & 0 & 0.8 & 0.5 \end{bmatrix}$$

The composition(bileşim, oluşum) of fuzzy relations P and Q

- Given fuzzy relations $P \subseteq X \times Y$, $Q \subseteq Y \times Z$
  - Composition on P and Q = $P \circ Q = R \subseteq X \times Z$
  - The membership degree of a chain <*x*, *y*, *z*> is determined by the degree of the weaker of the two links, <*x*, *y*> and <*y*, *z*> .

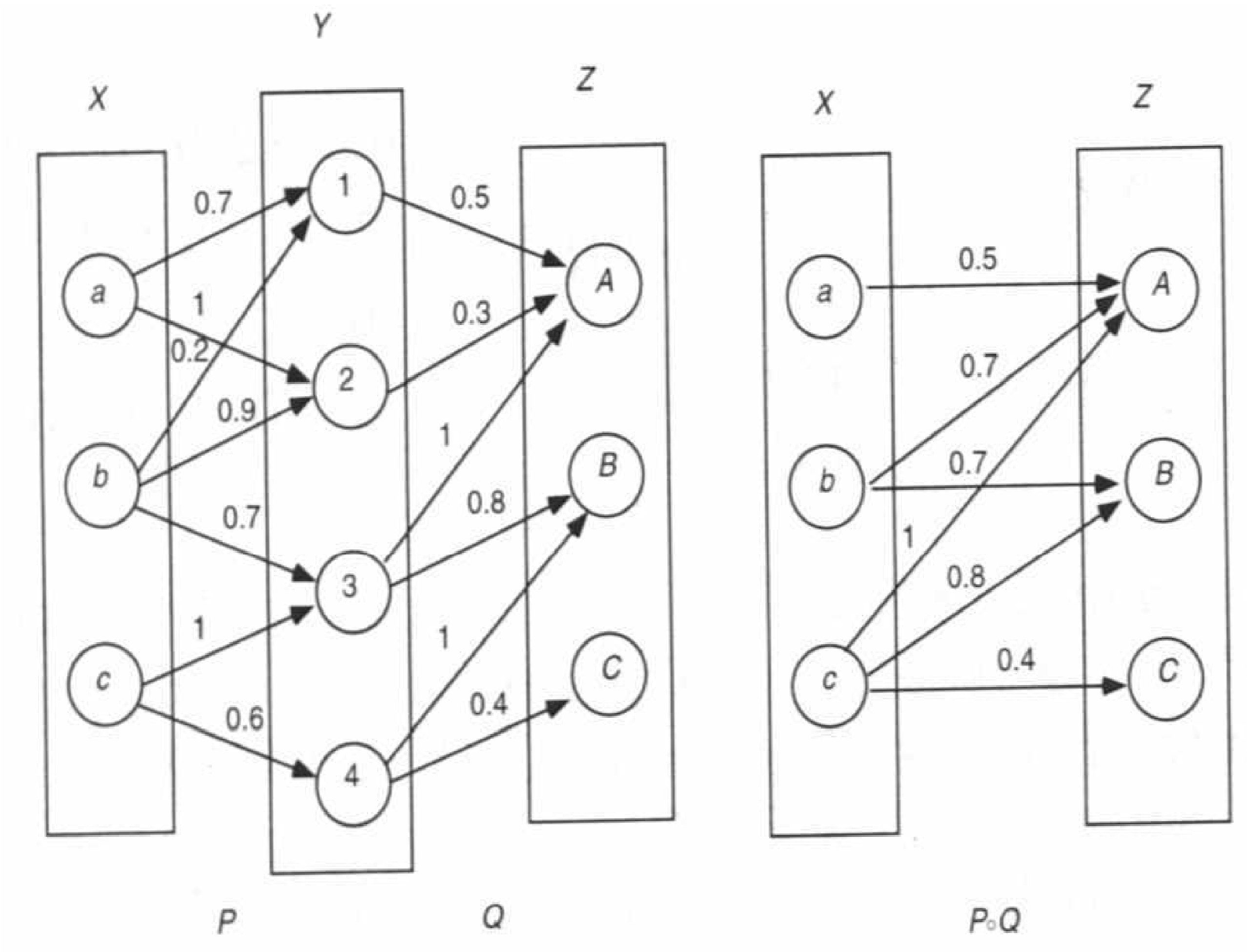
  - $R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min [\, P(x, y), Q(y, z)\,]$
  - $R(x, z) = (P \circ Q)(x, z) = \sup_{y \in Y} \min [\, P(x, y), Q(y, z)\,]$

Example (composition of fuzzy relations)

X = {a,b,c}

Y = {1,2,3,4}

Z = {A,B,C}



Example (matrix composition)

$X = \{p_1, p_2, p_3, p_4\}$
$Y = \{s_1, s_2, s_3\}$
$Z = \{d_1, d_2, d_3, d_4, d_5\}$

$$\mathbf{P} = \begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.8 & 0 & 0 \\ 0.7 & 0.7 & 0.9 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.8 & 0 & 0 \\ 0.7 & 0.7 & 0.9 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.4 & 0 \\ 0.5 & 0.5 & 0.5 & 0.4 & 0.2 \\ 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.7 & 0.7 & 0.7 & 0.9 & 0.6 \end{bmatrix}$$

$P \subseteq X \times Y$ $\quad Q \subseteq Y \times Z$ $\quad R \subseteq X \times Z$

Fuzzy equivalence relations and compatibility relations

- Equivalence (eşdeğerlik, denklik): Any fuzzy relation R that satisfies reflexive, symmetric and transitive properties.
  - R is reflexive ⇔ $R(x, x) = 1$ for all $x \in X$.
  - R is symmetric ⇔ $R(x, y) = R(y, x)$ for all $x, y \in X$.
  - R is transitive ⇔ $R(x, z) \geq \max_{y \in Y} \min [\, R(x, y), R(y, z)]$ all $x, z \in X$.

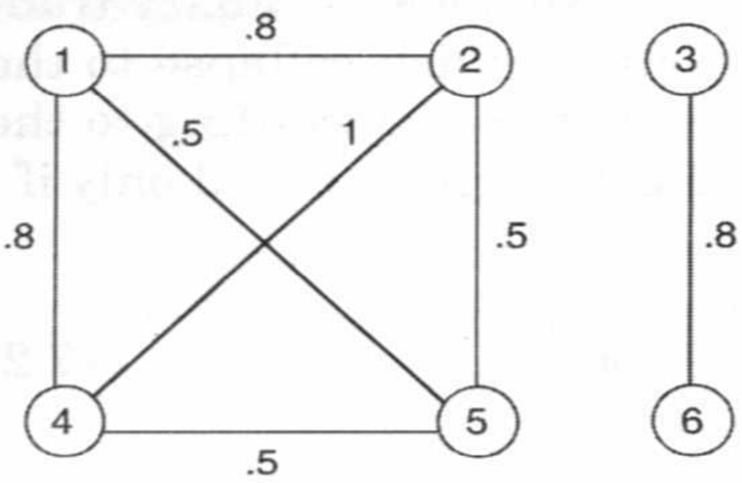
- Compatibility (bağdaşırlık, uyumluluk): satisfy reflexive and symmetric

fuzzy database modeling

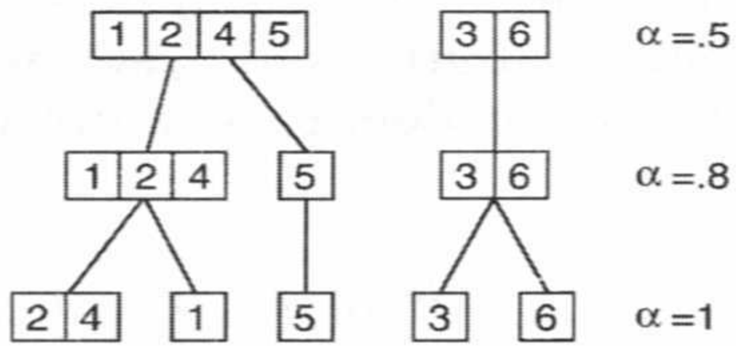
Example (fuzzy equivalence)

$$\mathbf{Q} = \begin{bmatrix} 1 & 0.8 & 0 & 0.8 & 0.5 & 0 \\ 0.8 & 1 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.8 \\ 0.8 & 1 & 0 & 1 & 0.5 & 0 \\ 0.5 & 0.5 & 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix}$$

$$R(1,4) \geq \max\{\min[R(1,2),R(2,4)], \min[R(1,5),R(5,4)]\}$$



Fuzzy equivalence relation Q

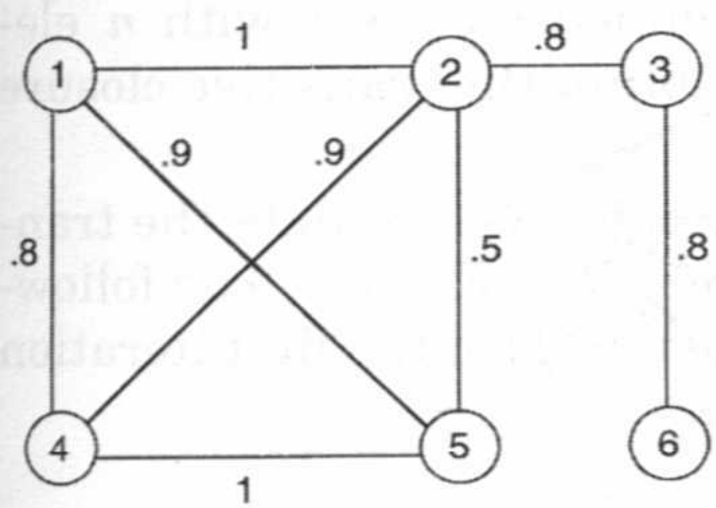


Equivalence classes in α-cuts of Q

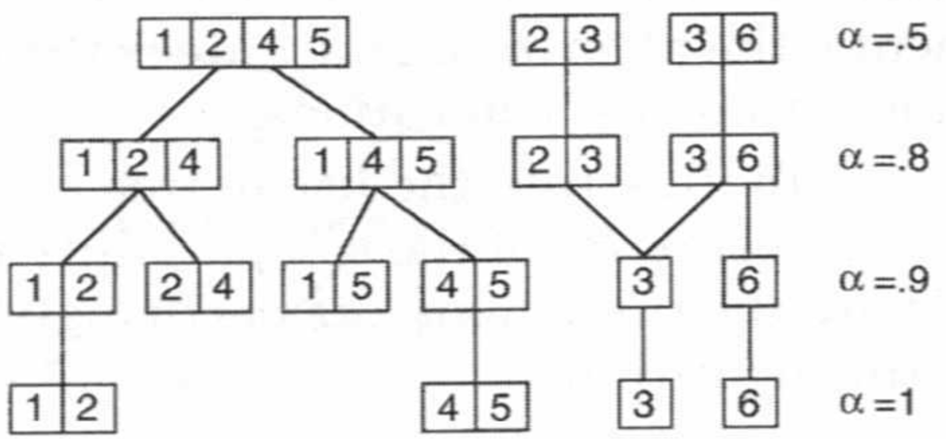
Example(fuzzy compatibility)

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 0 & 0.8 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.5 & 0 \\ 0 & 0.8 & 1 & 0 & 0 & 0.8 \\ 0.8 & 0.9 & 0 & 1 & 1 & 0 \\ 0.9 & 0.5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix}$$

$R(1,4) < \max \{\min[R(1,2),R(2,4)],$
$\min[R(1,5),R(5,4)]\}$



Fuzzy compatibility relation *R*

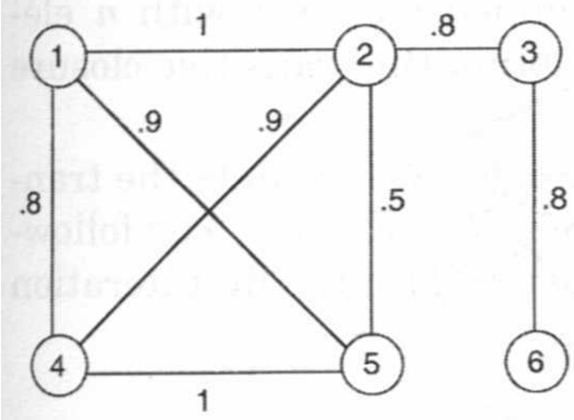


Maximal compatibility classes in α-cuts of *R*

Transitive closure

- The transitive closure of R is the smallest fuzzy relation that is transitive and contains R.
- interactive algorithm to find transitive closure R_T of R
  1. Compute R’ = $R \cup (R \circ R)$
  2. If $R' \neq R$, rename R’ as R and go to step 1. Otherwise, $R' = R_T$.

Evaluate the algorithm



Fuzzy compatibility relation R

$$\begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0.8 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 & 1 \end{bmatrix}$$

second iteration

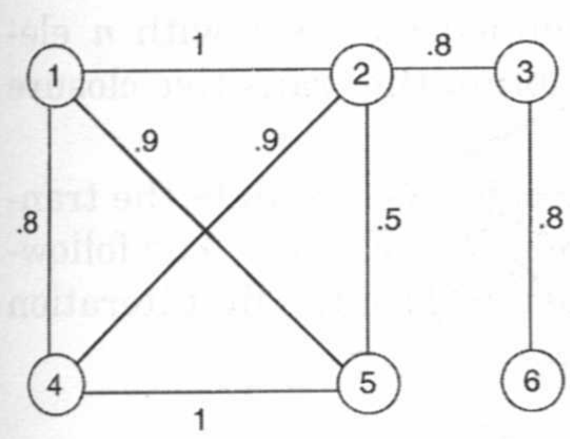
$$\begin{bmatrix} 1 & 1 & 0 & 0.8 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.5 & 0 \\ 0 & 0.8 & 1 & 0 & 0 & 0.8 \\ 0.8 & 0.9 & 0 & 1 & 1 & 0 \\ 0.9 & 0.5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0 \\ 0.9 & 0.9 & 0.5 & 1 & 1 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0.8 & 0.9 & 0.9 & 0 \\ 1 & 1 & 0.8 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.5 & 0.8 \\ 0.9 & 0.9 & 0.8 & 1 & 1 & 0 \\ 0.9 & 0.9 & 0.5 & 1 & 1 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 1 \end{bmatrix}$$

$R \qquad R \circ R \longleftrightarrow R \cup (R \circ R)$

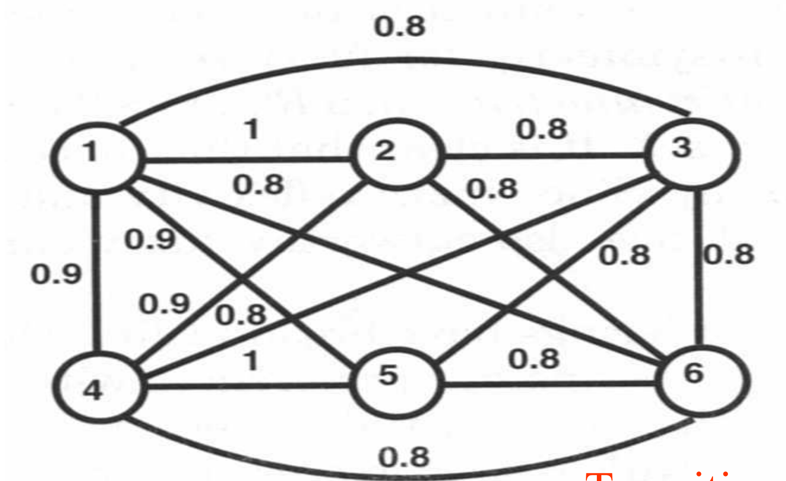
First iteration

equal

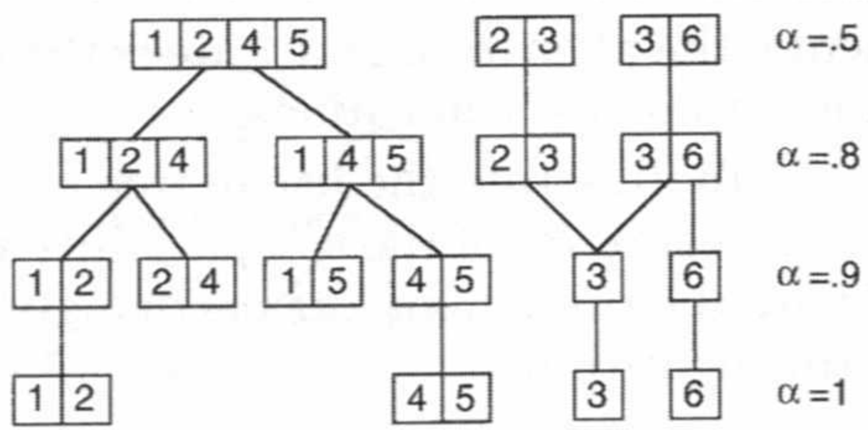
Example (transitive closure)

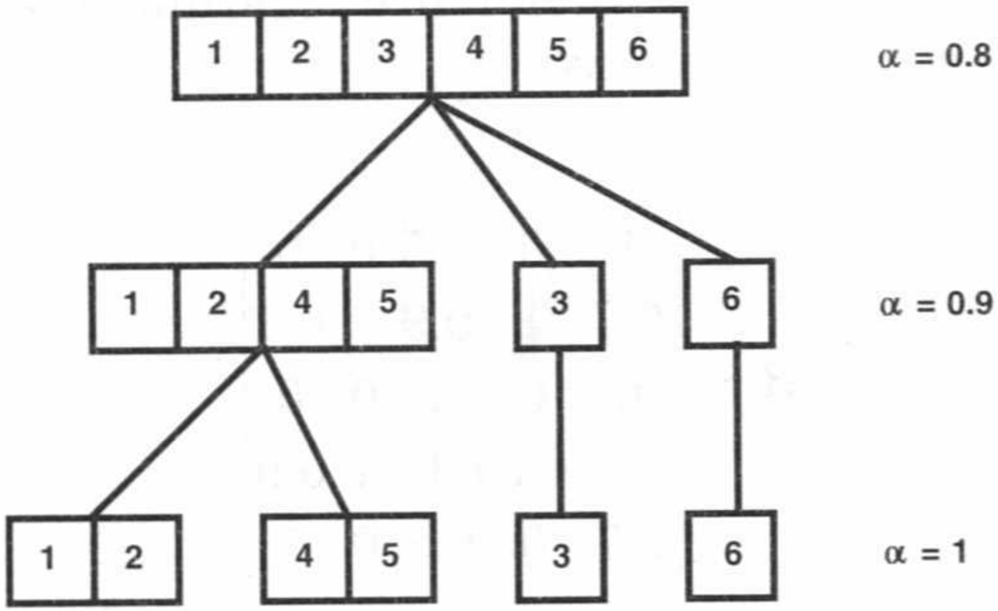


Fuzzy compatibility relation *R*



Transitive closure of R





Evaluate the algorithm (Example)

$$\mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .4 & 0 & 0 \\ 0 & 0 & .8 & 0 \end{bmatrix}$$

not equal

First iteration

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & .4 \\ 0 & .4 & 0 & 0 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & 0 & .5 \\ 0 & 0 & .8 & 1 \\ 0 & .4 & 0 & .4 \\ 0 & .4 & .8 & 0 \end{bmatrix} = \mathbf{R}'.$$

not equal

Second iteration

$$\mathbf{R} \circ \mathbf{R} = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .4 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .4 & .4 \end{bmatrix} \quad \mathbf{R} \cup (\mathbf{R} \circ \mathbf{R}) = \begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & .1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix} = \mathbf{R}'.$$

equal

Third iteration

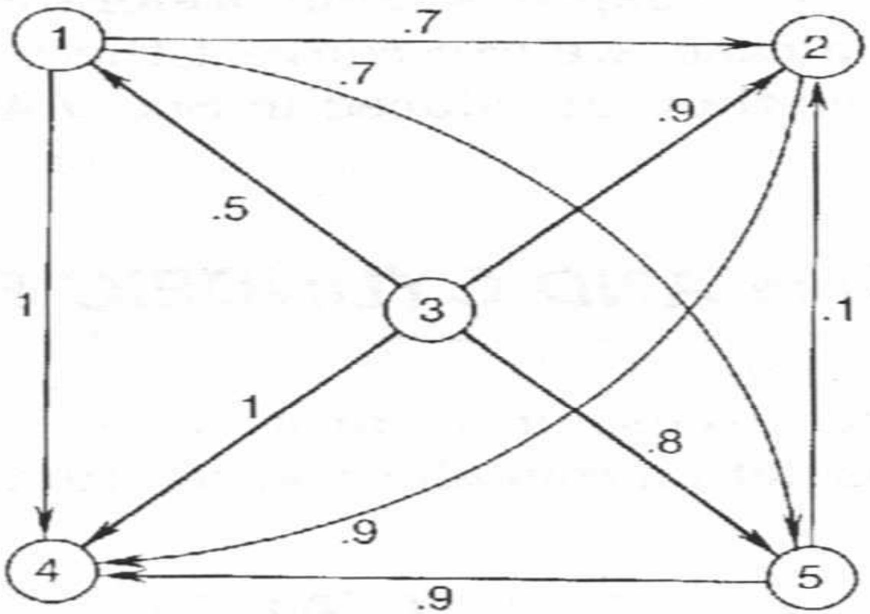
$$\begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & 1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix}$$

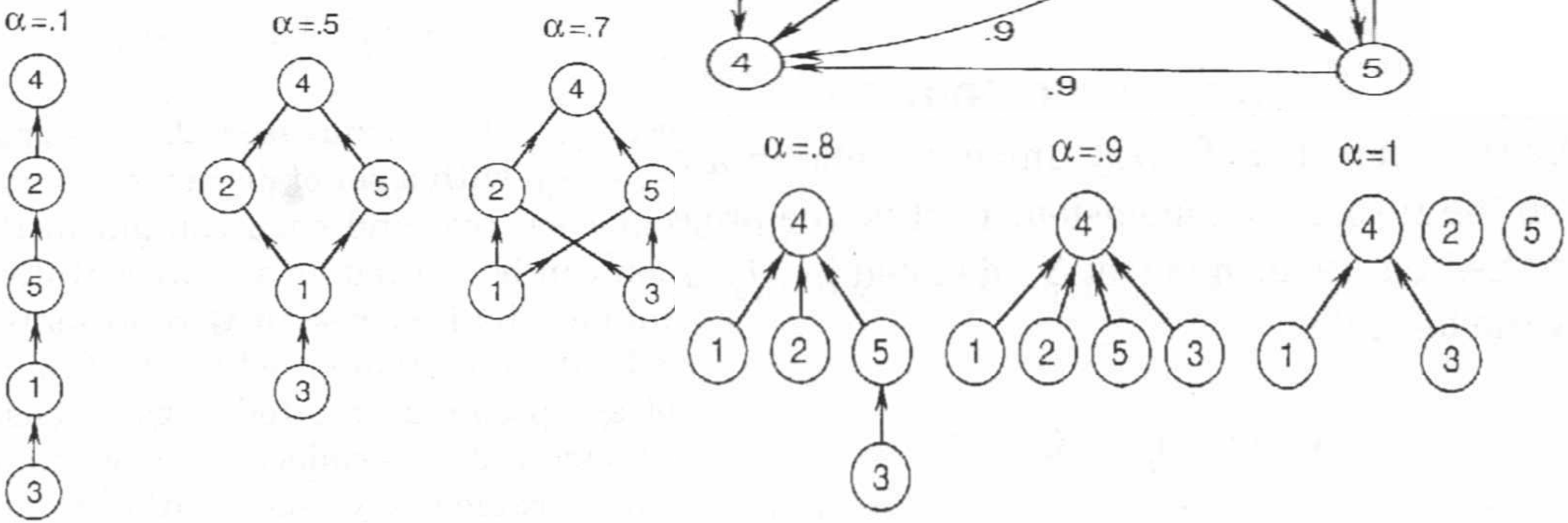
Fuzzy Partial ordering

- Fuzzy relations that satisfy reflexive, transitive and anti-symmetric, denoted by $x \leq y$
  - R on X is anti-symmetric ⇔ $R(x,y)>0$ and $R(y, x)>0$ imply that $x=y$ for any $x,y \in X$.

Example (fuzzy partial ordering)

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0.5 & 0 & 0 \\ 0.7 & 1 & 0.9 & 0 & 0.1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0.9 & 1 & 1 & 0.9 \\ 0.7 & 0 & 0.8 & 0 & 1 \end{bmatrix}$$





Projections example

$S = \{s_1, s_2, s_3\}$ symptoms

$D = \{d_1, d_2, d_3, d_4, d_5\}$ diseases

$$\mathbf{Q} = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix}$$

Projection on S

$$Q_1(s) = \max_{d \in D} Q(s, d)$$

Projection on D

$$Q_2(s) = \max_{s \in S} Q(s, d)$$

$$Q_1 = 0.7/s_1 + 0.8/s_2 + 0.9/s_3$$

$$Q_2 = 0.7/d_1 + 0.7/d_2 + 0.8/d_3 + 0.9/d_4 + 0.6/d_6$$

Projections

Given an n-dimensional fuzzy relation R on $X = X_1 \times X_2 \times \ldots \times X_n$ and any subset P(any chosen dimensions) of X, the projection R_P of R on P for each $p \in P$ is

$$R_P(p) = \max_{\overline{p} \in \overline{P}} R(p, \overline{p})$$

Where $\overline{P}$ is the remaining dimensions

Cylindric extension

Given an n-dimensional fuzzy relation R on $X = X_1 \times X_2 \times \ldots \times X_n$, any $(n+k)$-dimensional relation whose projection into the n dimensions of R yields R is called an extension of R.

An extension of R with respect to $Y = Y_1 \times Y_2 \times \ldots \times Y_h$ is call the cylindric extension ${}^{EY}R$ of R into Y

$$ {}^{EY}R(x,y) = R(x) $$

For all $x \in X$ and $y \in Y$.

Example (Cylindric extension)

$$Q_1 = 0.7/s_1 + 0.8/s_2 + 0.9/s_3$$

$$Q_2 = 0.7/d_1 + 0.7/d_2 + 0.8/d_3 + 0.9/d_4 + 0.6/d_6$$

$$^{ED}\mathbf{Q_1} = \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix}, \quad ^{ES}\mathbf{Q_2} = \begin{bmatrix} 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \end{bmatrix}$$

$$^{ED}\mathbf{Q_1} \cap {}^{ES}\mathbf{Q_2} = \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.8 & 0.6 \\ 0.7 & 0.7 & 0.8 & 0.9 & 0.6 \end{bmatrix} \supseteq Q = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0.6 \\ 0.5 & 0.5 & 0.8 & 0.4 & 0 \\ 0 & 0.7 & 0.2 & 0.9 & 0 \end{bmatrix}$$

Example of cylindric closure

- cylindric closure : the intersection of the cylindric extensions of R is the original relation R.

$$\mathbf{R} = \begin{bmatrix} 0.3 & 0.4 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.8 & 0.9 \end{bmatrix}$$

$X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$

$$R_1 = 0.4/x_1 + 0.7/x_2 + 0.2/x_3 + 0.9/x_4$$
$$R_2 = 0.3/y_1 + 0.5/y_2 + 0.8/y_3 + 0.9/y_4$$

min

$$^{EY}\mathbf{R_1} = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.4 \\ 0.7 & 0.7 & 0.7 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.9 & 0.9 & 0.9 & 0.9 \end{bmatrix} \quad ^{EX}\mathbf{R_2} = \begin{bmatrix} 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \\ 0.3 & 0.5 & 0.8 & 0.9 \end{bmatrix}$$